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Some mathematical properties of two versions of pseudo-differential operators involving the coupled fractional Fourier transform

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Abstract

The Pseudo-differential operators (p.d.o.) $A(x,y,D'_{x,y})$ and $\mathscr{A}(x,y,D'_{x,y})$ involving the coupled fractional Fourier transform $\mathscr{F}_{\alpha_1,\alpha_2}$ are defined. The symbol class $\Lambda(\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}))$ is discussed. We conclude the manuscript by applying some of the results to obtain some inequalities. At the end, boundedness of the Pseudo-differential operators are also discussed

1 Introduction and motivation

The fractional Fourier transform was developed in 1980 by Namias [1] as a means of determining the solutions to certain differential equations that sometimes arise in quantum physics. McBride and Kerr [2] further refined his findings by creating an operational calculus for the fractional Fourier transform. Fractional Fourier transform has drawn increased attention in recent years due to its many applications in the fields of image processing, signal analysis, and optics. This transformation is crucial for resolving a number of issues in signal processing, optics, and quantum physics [1, 3, 4, 5, 6, 7, 8, 5, 9, 10, 11]. A variety of mathematical analytic fields have examined the fractional Fourier transform, which is a generalisation of the ordinary Fourier transform. These areas include wavelets [12, 13], pseudo-differential operators [14], and generalised functions [15, 10, 16, 10]. The well-known Fourier transform of a function $\phi\in L_1(\mathbb{R})$, represented by $\widehat{\phi}$, is described as

$$\widehat{\phi}(\eta)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{i\eta\zeta}\phi(\zeta)d\zeta$$

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so that its inverse is given by

$$\phi(\zeta)=\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\eta\zeta}\widehat{\phi}(\eta)d\eta$$

provided the integrals exist.
We recall the one-dimensional fractional Fourier transform [4, 16, 17, 18, 16, 16, 19, 20, 21, 22, 23, 24, 23] of a function $\phi \in L_1(\mathbb{R})$ with parametre α, denoted by $(\mathscr{F}_\alpha\phi)(\eta)=\widehat{\phi}_\alpha(\eta)$ is given in $L_1(\mathbb{R})$ as follows:

$$(\mathscr{F}_\alpha\phi)(\eta)=\widehat{\phi}_\alpha(\eta)=\int_{\mathbb{R}} K_\alpha(\zeta,\eta)\phi(\zeta)d\zeta \tag{1}$$

where the kernel $K_\alpha(\zeta,\eta)$ is given by

$$K_\alpha(\zeta,\eta)=\begin{cases} C_\alpha e^{\frac{i(\zeta^2+\eta^2)\cot\alpha}{2}-i\zeta\eta\csc\alpha}, & \alpha\neq n\pi, n\in\mathbb{Z}\\ \frac{1}{\sqrt{2\pi}}e^{-i\zeta\eta}, & \alpha=\frac{\pi}{2}\\ \delta(\zeta-\eta), & \alpha=2n\pi\\ \delta(\zeta+\eta), & \alpha=(2n+1)\pi, \end{cases}$$

$C_\alpha=\sqrt{\frac{1-i\cot\alpha}{2\pi}}$ and studied some properties of this transform.
The corresponding inversion formula of $(\mathscr{F}_\alpha\phi)(\eta)$ is defined in the following ways

$$\phi(\zeta)=\int_{\mathbb{R}}\overline{K_\alpha(\zeta,\eta)}(\mathscr{F}_\alpha\varphi)(\eta)d\eta \tag{2}$$

$$\overline{K_\alpha}(\zeta,\eta)=\overline{C_\alpha}e^{\frac{-i(\zeta^2+\eta^2)\cot\alpha}{2}+i\zeta\eta\csc\alpha}$$

and $\overline{C_\alpha}=\sqrt{\frac{1+i\cot\alpha}{2\pi}}=C_{-\alpha}$.
Hence, $\overline{K_\alpha}(\zeta,\eta)=K_{-\alpha}(\zeta,\eta)$.
It implies that the inverse of a FrFT with the parameter α is the FrFT with the parameter $-\alpha$.
Exploiting the tensor product of n copies of the one-dimensional fractional Fourier transform each of order $\alpha_p,\ p=1,2,3,\dots,n$ [17], the fractional Fourier transform has been extended to the higher-dimensional transform.
We assume that $\alpha=(\alpha_1,\alpha_2), \mathbf{x}=(x,\eta), \mathbf{y}=(y,\zeta), K_\alpha(\mathbf{x},\mathbf{y})=K_{\alpha_1}(x,\eta).K_{\alpha_2}(y,\zeta)=K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)$, where $K_{\alpha_1}(x,\eta)$ and $\mathscr{K}_{\alpha_2}(y,\zeta)$ defined as above.
The two-dimensional fractional Fourier transform [25, 26, 27] is defined as follows:

$$\begin{aligned}[\mathscr{F}_\alpha\phi](\eta,\zeta)=[\mathscr{F}_{\alpha_1,\alpha_1}\phi](\eta,\zeta) &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_\alpha(\mathbf{x},\mathbf{y})\phi(x,y)dxdy\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1}(x,\eta)K_{\alpha_2}(y,\zeta)\phi(x,y)dxdy\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)\phi(x,y)dxdy. \quad (3)\end{aligned}$$

The corresponding inversion formula of (3) is defined as follows:

$$\phi(x,y)=\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)}[\mathscr{F}_{\alpha_1,\alpha_1}\phi](\eta,\zeta)d\eta d\zeta. \tag{4}$$

It is easy to observe that for $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, the two-dimensional fractional Fourier transform $\mathscr{F}_{\alpha_1,\alpha_2}$ becomes a classical two-dimensional Fourier transform.

Definition 1. *A tempered distribution ϕ belongs to the Sobolev type spce $\mathscr{H}^s(\mathbb{R}\times\mathbb{R})$, and $s\in\mathbb{R}$ if its coupled fractional Fourier transform $\mathscr{F}_{\alpha_1,\alpha_2}\phi$ corresponding to a locally integrable function $(\mathscr{F}_{\alpha_1,\alpha_2}\phi)(\xi,\eta)$ over $\mathbb{R}\times\mathbb{R}$ such that*

$$\|\phi\|_s = \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\{(1+|\xi|^2)(1+|\eta|^2)\}^{\frac{s}{2}}(\mathscr{F}_{\alpha_1,\alpha_2}\phi)(\xi,\eta)\right|^2 d\eta\, d\xi\right)^{\frac{1}{2}} < \infty. \tag{5}$$

This space is complete with respect to the norm $\|\phi\|_s$.

Definition 2. *The space $\mathscr{S}(\mathbb{R}\times\mathbb{R})$ is the collection of all complex valued infinitely differentiable functions $\phi(\xi,\eta)\in\mathbb{R}\times\mathbb{R}$ for every choice of $l_1,\ l_2,\ m_1,\ m_2\in\mathbb{N}_0$ which for*

$$\Gamma^{l_1,\,l_2}_{m_1,\,m_2}(\phi) = \sup_{(x,y)\in\mathbb{R}\times\mathbb{R}}\left|x^{l_1}y^{l_2}\frac{\partial^{m_1}}{\partial x^{m_1}}\frac{\partial^{m_2}}{\partial y^{m_2}}\phi(x,y)\right| < \infty. \tag{6}$$

The dual of $\mathscr{S}(\mathbb{R}\times\mathbb{R})$ is denoted by $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$.
If φ is a locally integrable and polynomial growth function on $\mathbb{R}\times\mathbb{R}$, then φ generates a distribution in $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ as follows:

$$\langle\varphi,\ \phi\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(\xi,\eta)\phi(\xi,\eta)d\xi d\eta,\quad \forall\phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R}). \tag{7}$$

The elements of $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ are known as tempered distributions.

Theorem 1. *Let $K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)$ be the kernel of the two-dimensional fractional Fourier transform. Then, for all $\varphi(x,y)\in\mathscr{S}(\mathbb{R}\times\mathbb{R})$, we have*
(i) $D^r_{x,y}K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta) = \{i(\eta\csc\alpha_1+\zeta\csc\alpha_2)\}^r K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)$,
(ii) $\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(x,y)D^r_{x,y}K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)dxdy = \int_{\mathbb{R}}\int_{\mathbb{R}}K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)(D'_{x,y})^r\varphi(x,y)dxdy$,
(iii) $\mathscr{F}_{\alpha_1,\alpha_2}\{(D'_{x,y})^r\varphi(x,y)\}(\eta,\zeta) = \{i(\eta\csc\alpha_1+\zeta\csc\alpha_2)\}^r(\mathscr{F}_{\alpha_1,\alpha_2}\varphi(x,y))(\eta,\zeta)$,
for all $r\in\mathbb{N}$, where $D_{x,y} = [\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+i(x\cot\alpha_1+y\cot\alpha_2)]$ and $D'_{x,y} = -[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}-i(x\cot\alpha_1+y\cot\alpha_2)]$.

Proof. See [25]. □

2 Symbol Classes

Let $a(x,y,\xi,\zeta)$ be a complex valued function defined for $x,y,\xi\neq 0,\zeta\neq 0\in\mathbb{R}$. The function $a(x,y,\xi,\zeta)\in\mathbb{C}^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{R}-\{0\}\times\mathbb{R}-\{0\})$ is said to be an element of the class Λ if and only if $a(x,y,t_1\xi,t_2\zeta) = a(x,y,\xi,\zeta)$ for $t_1>0,\ t_2>0$, and assume also that

$$\lim_{(|x|,|y|)\to(\infty,\infty)} = a(\infty,\infty,\xi,\zeta)$$

exists for $\xi \neq 0, \zeta \neq 0 \in \mathbb{R}$ and $a(\infty,\infty,\xi,\zeta)$ is a mapping $\mathbb{C}^{\infty}$-function.

Now we define $a'(x,y,\xi,\zeta) = a(x,y,\xi,\zeta) - a(\infty,\infty,\xi,\zeta)$, and assume the estimates

$$(1+x^2+y^2)^p \left| \frac{\partial^k}{\partial x^k}\frac{\partial^l}{\partial y^l}\frac{\partial^m}{\partial \xi^m}\frac{\partial^n}{\partial \zeta^n} a'(x,y,\xi,\zeta) \right| \leq \mathbb{C}_{p,k,l,m,n}, \quad \forall x,\ y,\ \xi \neq 0,\ \zeta \neq 0 \in \mathbb{R} \tag{8}$$

here p=1,2,3,.....,k, l, m, n are natural numbers.

Theorem 2. *(i) We get* $|a(\infty,\infty,\xi,\zeta) - a(\infty,\infty,\delta,\eta)| \leq \mathbb{C}\big((|\xi-\delta| + |\zeta-\eta|)/(|\xi|+|\zeta|+|\delta|+|\eta|)\big)$, $\forall \xi,\ \zeta,\ \delta,\ \eta$ *arbitray in* $\mathbb{R}-\{0\}$.

(ii) The estimates $(1+x^2csc^2\alpha_1 + y^2csc^2\alpha_2)^p|\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)| \leq \mathbb{M}_p$, $\forall x,\ y,\ \xi \neq 0,\ \zeta \neq 0 \in \mathbb{R},\ p = 1,\ 2,\ 3,\ 4,\ 5...;$

(iii) $(1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p|\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta) - \mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\delta,\eta)| \leq \mathbb{M}_p(|\xi-\delta|+|\zeta-\eta|)(|\xi|+|\zeta|+|\delta|+|\eta|)^{-1}$, $\forall \xi,\ \zeta,\ \delta,\ \eta \in \mathbb{R}-\{0\}$, $\forall x,y \in \mathbb{R}$, $p = 1,2,...$*to* ∞ *being*

$$\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,x,y)a'(t,u,\xi,\zeta)dtdu, \quad \forall x,y,\xi \neq 0, \zeta \neq 0 \in \mathbb{R}$$

are verified.

Proof. (i) Similar proof of Theorem 1 (a)[28].
(ii) We get the equality

$$\begin{aligned} & (1+x^2csc^2\alpha_1 + y^2csc^2\alpha_2)^p \mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta) \\ = \; & \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,x,y)(I-D'_{x,y})^p a'(t,u,\xi,\zeta)dtdu, \\ & x,y,\xi \neq 0, \zeta \neq 0 \in \mathbb{R},\ D'_{x,y} = -\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - i(xcot\alpha_1 + ycot\alpha_2)\right] \end{aligned} \tag{9}$$

and therefore is verified the estimate

$$\begin{aligned} & \left|(1+x^2csc^2\alpha_1 + y^2csc^2\alpha_2)^p \mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)\right| \\ \leq \; & C_{\alpha_1}C_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+t^2csc^2\alpha_1 + u^2csc^2\alpha_2)^q|(I-D'_{x,y})^p a'(t,u,\xi,\zeta)| \\ & \times(1+t^2csc^2\alpha_1 + u^2csc^2\alpha_2)^{-q}dtdu \\ \leq \; & C_{\alpha_1}C_{\alpha_2}C_1\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{(1+t^2csc^2\alpha_1 + u^2csc^2\alpha_2)^q}dtdu = \mathbb{M}_p \end{aligned} \tag{10}$$

for q sufficient large.

(iii) We get

$$\begin{aligned}
& (1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p|\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)-\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\delta,\eta)| \\
= & \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,x,y)(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^q(I-D'_{x,y})^p \\
\times & \left[a'(t,u,\xi,\zeta)-a'(t,u,\delta,\eta)\right](1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^{-q}dtdu.
\end{aligned}$$

Let us put now

$$b_{p,q}(t,u,\xi,\zeta)=(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^q(I-D'_{x,y})^pa'(t,u,\xi,\zeta). \qquad (11)$$

We obtain then the estimate

$$\begin{aligned}
& (1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p\left|\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)-\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\delta,\eta)\right| \\
\leq & |C_{\alpha_1}C_{\alpha_2}|\int_{\mathbb{R}}\int_{\mathbb{R}}\left|b_{p,q}(t,u,\xi,\zeta)-b_{p,q}(t,u,\delta,\eta)\right| \\
& \times(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^{-q}dtdu. \qquad (12)
\end{aligned}$$

Consequently, it will be sufficient to show here that with a constant independent of $x,y\in\mathbb{R}$ we have, for $x,y\in\mathbb{R},\ \xi,\ \zeta,\ \delta,\ \eta\in\mathbb{R}-\{0\}$, the estimate

$$\left|b_{p,q}(t,u,\xi,\zeta)-b_{p,q}(t,u,\delta,\eta)\right|\leq D_{\alpha_1,\alpha_2}(|\xi-\delta|+|\zeta-\eta|)(|\xi|+|\zeta|+|\delta|+|\eta|)^{-1}. \qquad (13)$$

It can be easily proved from (ii), (12)and (13). □

The term "pseudo-differential operators"[29, 30, 31, 32] has a fairly broad definition and covers such topics as harmonic analysis, partial differential equation, geometry, mathematical physics, microlocal analysis, time-frequency analysis, imaging, computations, and quantum mechanics. In mathematics, natural sciences, medicine, scientific computing, and engineering, current trends and novel applications are highlighted. The emphasis is on contemporary developments in different branches of engineering, mathematical sciences, the natural sciences, medicine, scientific computers.

In reality, Kohn-Nirenberg and Hörmander [33] were the ones who first introduced the pseudo-differential calculus, and later authors expanded on it, primarily in a local context, to examine local regularity and local solvability of PDEs.

Pseudo-differential operators on $\mathbb{R}_+$ are standard or conventional generalizations of partial differential operators or ordinary differential operators and singular integrals.

Many faculties, scientists, Ph.D students and researchers of other field developed the theory of pseudo-differential operators with the help of different types of integral operators like Fourier transforms (see [34, 28]), Hankel transform (see [35, 36, 37]), Fourier Bessel Transform on $\mathbb{R}_+$ (see [19, 20]), Weinstein transform (see [38]), Laguerre hypergroups (see [39]) and Jacobi differential operators (see [40]).

3 Pseudo-Differential Operator $A(x,y,D'_{x,y})$ related to $\mathscr{F}_{\alpha_1,\alpha_2}$

Let $a(x,y,\xi,\zeta) = a'(x,y,\xi,\zeta) + a(\infty,\infty,\xi,\zeta)$ be a symbol, and, as previously,

$$\mathscr{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,x,y)a'(t,u,\xi,\zeta)dtdu, \ \ \forall x,y,\xi \neq 0, \zeta \neq 0 \in \mathbb{R}.$$

Let us define, for any $\phi \in S^{(}(\mathbb{R}\times\mathbb{R})$ and $x,y \in \mathbb{R}$, a function $\mu(x,y) = (A(x,y,D'_{x,y})\phi)(x,y)$, by

$$(A(x,y,D'_{x,y})\phi)(x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,x,y)G_{\alpha_1,\alpha_2}(t,u)dtdu, \tag{14}$$

where the function $G_{\alpha_1,\alpha_2}(t,u)$ is given by

$$G_{\alpha_1,\alpha_2}(t,u) = a(\infty,\infty,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u) + \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{a'}_{\alpha_1,\alpha_2}(t-\xi,u-\eta,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)d\xi d\eta. \tag{15}$$

Evidently, it has to be proved that $G_{\alpha_1,\alpha_2}(t,u)$ is the Coupled fractional Fourier transformable, in fact, we have $G_{\alpha_1,\alpha_2}(t,u) \in L_1(\mathbb{R}\times\mathbb{R})$ as

$$\left|a(\infty,\infty,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)\right| \leq \max_{|t|=1,|u|=1} |a(\infty,\infty,t,u)||\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)| \in L_1(\mathbb{R}\times\mathbb{R}),$$

then obviously, it is sufficient to show that

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\widehat{a'}_{\alpha_1,\alpha_2}(t-\xi,u-\eta,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)\right| d\xi d\eta dtdu < \infty;$$

we have in fact, $\forall p = 1,2,3,...$

$$\begin{aligned} & \int_{\mathbb{R}}\int_{\mathbb{R}} \left|\widehat{a'}_{\alpha_1,\alpha_2}(t-\xi,u-\eta,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)\right| d\xi d\eta \\ \leq \ & \mathbb{M}_p \int_{\mathbb{R}}\int_{\mathbb{R}} (1+|t-\xi|^2 csc^2\alpha_1 + |u-\eta|^2\csc^2\alpha_2)^{-p}|\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)|d\xi d\eta. \end{aligned}$$

This last expression is the convolution between $(1+|t|^2 csc^2\alpha_1 + |u|^2\csc^2\alpha_2)^{-p}$ and $|\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)|$ both integrable for p sufficiently large.
Hence

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\widehat{a'}_{\alpha_1,\alpha_2}(t-\xi,u-\eta,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)\right| d\xi d\eta dtdu < \infty.$$

Thus $A(x,y,D'_{x,y})\phi$ is continuous and bounded on $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$. Hence we can say that

$$\begin{aligned} \left[\mathscr{F}_{\alpha_1,\alpha_2}\big(A(x,y,D'_{x,y})\big)\phi\right](t,u) &= a(\infty,\infty,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u) \\ &\quad + \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{a'}_{\alpha_1,\alpha_2}(t-\xi,u-\eta,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)d\xi d\eta \end{aligned}$$

is verified the Coupled fractional Fourier transform in $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$.

Theorem 3. *If $a(x,y,\xi,\zeta)$ is a symbol, we have*

$$\begin{aligned}[A(x,y,D'_{x,y})\phi](x,y) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{K_{\alpha_1,\alpha_2}(x,y,\xi,\zeta)} \\ &\quad \times\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,\xi,\zeta)a(x,y,\xi,\zeta)\phi(t,u)dtdu\right\}d\xi d\zeta\end{aligned}$$

for every $\phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R})$, x, $y \in \mathbb{R}$.

Proof. It will be sufficient to prove that
(i) The integral $\int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,\xi,\zeta)a(x,y,\xi,\zeta)\phi(t,u)dtdu$ is absolutely covergent.
(ii) We have $G_{\alpha_1,\alpha_2}(\xi,\zeta) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,\xi,\zeta)a(x,y,\xi,\zeta)\phi(t,u)dtdu.$
We have (i), in fact, as $a(x,y,\xi,\zeta) = a(\infty,\infty,\xi,\zeta) + e^{\frac{i}{2}(x^2\cot\alpha_1 + y^2\cot\alpha_2)}a'(x,y,\xi,\zeta)$.
It is sufficient to prove the absolute convergence of

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,\xi,\zeta)a(\infty,\infty,\xi,\zeta)\phi(t,u)dtdu \\ = \;& a(\infty,\infty,\xi,\zeta)\int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,\xi,\zeta)\phi(t,u)dtdu \\ = \;& a(\infty,\infty,\xi,\zeta)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\zeta) \quad \text{for } \phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R})\end{aligned}$$

and

$$\int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,\xi,\zeta)e^{\frac{i}{2}(t^2\cot\alpha_1 + u^2\cot\alpha_2)}a'(t,u,\xi,\zeta)\phi(t,u)dtdu \quad \text{for } \phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R}).$$

As

$|e^{\frac{i}{2}(t^2\cot\alpha_1 + u^2\cot\alpha_2)}a'(t,u,\xi,\zeta)| \leq \mathbb{M}_p(1+|t|^2+|u|^2)^{-p}$ for every p, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,\xi,\zeta)e^{\frac{i}{2}(t^2\cot\alpha_1 + u^2\cot\alpha_2)}a'(t,u,\xi,\zeta)\phi(t,u)dtdu\right| \\ \leq \;& \int_{\mathbb{R}}\int_{\mathbb{R}} |K_{\alpha_1,\alpha_2}(t,u,\xi,\zeta)||e^{\frac{i}{2}(x^2\cot\alpha_1 + y^2\cot\alpha_2)}a'(x,y,\xi,\zeta)||\phi(t,u)|dtdu \\ \leq \;& \frac{\mathbb{M}_p}{2\pi}\frac{1}{\sqrt{sin\alpha_1 sin\alpha_2}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\phi(t,u)|}{(1+|t|^2+|u|^2)^p}dtdu.\end{aligned}$$

It implies that $\int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,\xi,\zeta)a(t,u,\xi,\zeta)\phi(t,u)dtdu$ is absolutely covergent.
In order to prove (ii), it is sufficient that

$$\begin{aligned}&\overline{C_{\alpha_1}C_{\alpha_2}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda_1^2)cot\alpha_1 + i(u\lambda_2-\lambda_2^2)cot\alpha_2}\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u) \\ &\times\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 \\ = \;& \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,\tau,\zeta)e^{\frac{i}{2}(\tau^2\cot\alpha_1 + \zeta^2\cot\alpha_2)}a'(\tau,\zeta,t,u)\phi(\tau,\zeta)d\tau d\zeta.\end{aligned}$$

Now, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,\tau,\zeta)e^{\frac{i}{2}(\tau^2\cot\alpha_1+\zeta^2\cot\alpha_2)}a'(\tau,\zeta,t,u)\phi(\tau,\zeta)d\tau d\zeta \\
= \; & \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,\tau,\zeta)\Bigg[\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{K_{\alpha_1,\alpha_2}(x_1,y_1,\tau,\zeta)}e^{\frac{i}{2}(\tau^2\cot\alpha_1+\zeta^2\cot\alpha_2)} \\
& \times \widehat{a'}_{\alpha_1,\alpha_2}(x_1,y_1,t,u)dx_1dy_1\Bigg]\phi(\tau,\zeta)d\tau d\zeta \\
= \; & \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} C_{\alpha_1}C_{\alpha_2}\overline{C_{\alpha_1}C_{\alpha_2}}e^{\frac{i}{2}(t^2+\tau^2)cot\alpha_1-it\tau csc\alpha_1}e^{\frac{i}{2}(u^2+\zeta^2)cot\alpha_2-iu\zeta csc\alpha_2} \\
& \times e^{-\frac{i}{2}(x_1^2+\tau^2)cot\alpha_1+ix_1\tau csc\alpha_1}e^{-\frac{i}{2}(y_1^2+\zeta^2)cot\alpha_2+iy_1\zeta csc\alpha_2}\widehat{a'}_{\alpha_1,\alpha_2}(x_1,y_1,t,u)\phi(\tau,\zeta) \\
& \times dx_1dy_1d\tau d\zeta \\
= \; & \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} C_{\alpha_1}C_{\alpha_2}\overline{C_{\alpha_1}C_{\alpha_2}}e^{\frac{i}{2}[(t-x_1)^2+\tau^2]\cot\alpha_1-i\tau(t-x_1)csc\alpha_1} \\
& \times e^{\frac{i}{2}[(u-y_1)^2+\zeta^2]\cot\alpha_2-i\zeta(u-y_1)csc\alpha_2}e^{ix_1(t-x_1)cot\alpha_1+iy_1(u-y_1)cot\alpha_2} \\
& \times \widehat{a'}_{\alpha_1,\alpha_2}(x_1,y_1,t,u)\phi(\tau,\zeta)dx_1dy_1d\tau d\zeta.
\end{aligned}$$

By making in the internal integral the substitution $t-x_1=\lambda_1$ and $u-y_1=\lambda_2$

$$\begin{aligned}
= \; & \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} C_{\alpha_1}C_{\alpha_2}\overline{C_{\alpha_1}C_{\alpha_2}}e^{\frac{i}{2}[\lambda_1^2+\tau^2]\cot\alpha_1-i\tau\lambda_1csc\alpha_1}e^{\frac{i}{2}[\lambda_2^2+\zeta^2]\cot\alpha_2-i\zeta\lambda_2csc\alpha_2} \\
\times \; & e^{i(t\lambda_1-\lambda_1^2)cot\alpha_1+i(u\lambda_2-\lambda_2^2)cot\alpha_2}\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)\phi(\tau,\zeta)dx_1dy_1d\tau d\zeta. \\
= \; & \overline{C_{\alpha_1}C_{\alpha_2}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda_1^2)cot\alpha_1+i(u\lambda_2-\lambda_2^2)cot\alpha_2}\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u) \\
& \times \widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1d\lambda_2.
\end{aligned}$$

This completes the proof. □

Theorem 4. *We have the inequality*

$$\|A(x,y,D'_{x,y})\phi\|_s \leq C_{\alpha_1,\alpha_2}\|\phi\|_s, \tag{16}$$

$\forall s\in\mathbb{R},\ \phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R})$*, for a certain constant* C_{α_1,α_2} .

Proof. We have in fact the immediate decomposition

$$A(x,y,D'_{x,y}) = A(\infty,\infty,D'_{x,y}) + A'(x,y,D'_{x,y}).$$

We must remark that for $\phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R})$. We have by definition

$$\widehat{A}_{\alpha_1,\alpha_2}(\infty,\infty,D'_{x,y})\phi(t,u) = a(\infty,\infty,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u).$$

We have

$$\begin{aligned}
\left[\mathscr{F}_{\alpha_1,\alpha_2}\big(A'(x,y,D'_{x,y})\big)\phi\right](t,u) \;=\; & \overline{C_{\alpha_1}C_{\alpha_2}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda_1^2)cot\alpha_1+i(u\lambda_2-\lambda_2^2)cot\alpha_2} \\
& \times\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1d\lambda_2.
\end{aligned}$$

Then we see that first of all

$$\begin{aligned}\left\|A(\infty,\infty,D'_{x,y})\phi\right\|_s^2 &= \int_{\mathbb{R}}\int_{\mathbb{R}}(1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}|a(\infty,\infty,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)|^2 dtdu \\ &\le \left(\sup_{|t|=|u|=1}|a(\infty,\infty,t,u)|\right)^2\|\phi\|_s^2.\end{aligned}$$

Therefore

$$\left\|A(\infty,\infty,D'_{x,y})\phi\right\|_s \le \mathbf{C}_1\|\phi\|_s.$$

Less trivial is estimate for $A'(x,y,D'_{x,y})\phi$. Its coupled fractional Fourier transform in $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ equals

$$\overline{C_{\alpha_1}C_{\alpha_2}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda_1^2)cot\alpha_1+i(u\lambda_2-\lambda_2^2)cot\alpha_2}\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2$$

and then (using the definition of $\mathbb{H}^s(\mathbb{R}\times\mathbb{R})$). We will have to estimate the norm $L_2(\mathbb{R}\times\mathbb{R})$ of the expression

$$\begin{aligned}&(1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}\overline{C_{\alpha_1}C_{\alpha_2}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda_1^2)cot\alpha_1+i(u\lambda_2-\lambda_2^2)cot\alpha_2} \\ &\times\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2\end{aligned}$$

which is equal to $\mathscr{U}_s(t,u)$.
Now, we have

$$\begin{aligned}\left|\mathscr{U}_s(t,u)\right| &= |\overline{C_{\alpha_1}C_{\alpha_2}}|\left|\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}e^{i(t\lambda_1-\lambda_1^2)cot\alpha_1+i(u\lambda_2-\lambda_2^2)cot\alpha_2}\right. \\ &\quad\left.\times\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2\right| \\ &= |\overline{C_{\alpha_1}C_{\alpha_2}}|\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|t|^2)^{\frac{s}{2}}(1+|\lambda_1|^2)^{\frac{-s}{2}}(1+|u|^2)^{\frac{s}{2}}(1+|\lambda_2|^2)^{\frac{-s}{2}} \\ &\quad\times|\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)|(1+|\lambda_1|^2)^{\frac{s}{2}}(1+|\lambda_2|^2)^{\frac{s}{2}} \\ &\quad\times|\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)|d\lambda_1 d\lambda_2 \qquad (17) \\ &\le |\overline{C_{\alpha_1}C_{\alpha_2}}|\mathscr{D}_l 2^{|s|}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|t-\lambda_1|^2)^{\frac{|s|}{2}}(1+|u-\lambda_2|^2)^{\frac{|s|}{2}} \\ &\quad\times(1+|t-\lambda_1|^2\csc^2\alpha_1)^{\frac{-l}{2}}(1+|u-\lambda_2|^2\csc^2\alpha_2)^{\frac{-l}{2}}(1+|\lambda_1|^2)^{\frac{s}{2}} \\ &\quad\times(1+|\lambda_2|^2)^{\frac{s}{2}}|\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)|d\lambda_1 d\lambda_2 \qquad (18) \\ &= \overline{C_{\alpha_1}C_{\alpha_2}}|\mathscr{D}_l 2^{|s|}\int_{\mathbb{R}}\int_{\mathbb{R}} g_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2)f_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 \quad \text{(say)} \\ &= \overline{C_{\alpha_1}C_{\alpha_2}}|\mathscr{D}_l 2^{|s|}(g_{\alpha_1,\alpha_2}*f_{\alpha_1,\alpha_2})(t,u).\end{aligned}$$

If l is large, $g_{\alpha_1,\alpha_2}\in L_1(\mathbb{R}\times\mathbb{R})$. Also since $\widehat{\phi}_{\alpha_1,\alpha_2}\in\mathscr{S}(\mathbb{R}\times\mathbb{R}),\ f_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)\in L_2(\mathbb{R}\times\mathbb{R})$.
Then $g_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2)f_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)$ belongs to $L_2(\mathbb{R}\times\mathbb{R})$ and the inequality

$$\|g_{\alpha_1,\alpha_2}*f_{\alpha_1,\alpha_2}\|_{L_2(\mathbb{R}\times\mathbb{R})}\le\|g_{\alpha_1,\alpha_2}\|_{L_1(\mathbb{R}\times\mathbb{R})}\|f_{\alpha_1,\alpha_2}\|_{L_2(\mathbb{R}\times\mathbb{R})}.$$

It implies that

$$\|\mathscr{U}_s\|_{L_2(\mathbb{R}\times\mathbb{R})} \leq |C_2(\alpha_1,\alpha_2)|\|\phi\|_s.$$

Now

$$\begin{aligned}
\|A(\infty,\infty,D'_{x,y})\phi + A'(x,y,D'_{x,y})\phi\|_s &\leq \|A(\infty,\infty,D'_{x,y})\phi\|_s + \|A'(x,y,D'_{x,y})\phi\|_s \\
\|A(x,y,D'_{x,y})\phi\|_s &\leq \mathbf{C}_1\|\phi\|_s + |C_2(\alpha_1,\alpha_2)|\|\phi\|_s = C_{\alpha_1,\alpha_2}\|\phi\|_s.
\end{aligned}$$

This completes the proof. □

4 The pseudo-differential operator $\mathscr{A}(x,y,D'_{x,y})$

We consider a symbol $a(x,y,\xi,\zeta)$. We introduce an operator $\mathscr{A}(x,y,D'_{x,y})$ of $\mathscr{S}(\mathbb{R}\times\mathbb{R})$ in $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ by means of the formula

$$[\mathscr{A}(x,y,D'_{x,y})\phi](x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,x,y)\mathscr{H}_{\alpha_1,\alpha_2}(t,u)dtdu,$$

where, for $\phi\in\mathscr{S}$, the function $\mathscr{H}_{\alpha_1,\alpha_2}(t,u)$ is defined by the relation

$$\begin{aligned}
\mathscr{H}_{\alpha_1,\alpha_2}(t,u) &= a(\infty,\infty,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u) \\
&+ \overline{C_{\alpha_1}C_{\alpha_2}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda_1^2)cot\alpha_1+i(u\lambda_2-\lambda_2^2)cot\alpha_2} \\
&\quad \times\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2
\end{aligned}$$

$\forall\phi\in\mathscr{S}$ and $t\neq 0,\ u\neq 0\in\mathbb{R}$.
With the same proof used for $A(x,y,D'_{x,y})$ we have; the function $\mathscr{A}(x,y,D'_{x,y})$ is continuous and bounded for $x,y\in\mathbb{R}$. Besides, we see that if the symbol $a(x,y,t,u)$ does not depend on x,y, we have $A(D'_{x,y}) = \mathscr{A}(D'_{x,y})$.

Theorem 5. *We have*

$$\begin{aligned}
[\mathscr{A}(x,y,D'_{x,y})\phi](x,y) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{K_{\alpha_1,\alpha_2}(x,y,t,u)} && (19) \\
&\quad \times a(x,y,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)dtdu, \quad \forall\phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R}). && (20)
\end{aligned}$$

Proof. As $a(x,y,t,u) = a(\infty,\infty,t,u) + e^{\frac{i}{2}(x^2\cot\alpha_1+y^2\cot\alpha_2)}a'(x,y,t,u)$ and $\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)\in\mathscr{S}(\mathbb{R}\times\mathbb{R})$, the integral is absolutely convergent.
We have, then:

$$\int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}\overline{(x,y,\xi,\zeta)}\left[\overline{C_{\alpha_1}C_{\alpha_2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\widehat{a}_{\alpha_1,\alpha_2}(\xi-t,\zeta-u,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)e^{i(\xi t-t^2)\cot\alpha_1}e^{i(\zeta u-u^2)cot\alpha_2}dtdu\right]d\xi d\zeta \quad (21)$$

is absolutely convergent because

$$\begin{aligned}
& K_{\alpha_1,\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{(x,y,t,u)e^{\frac{i}{2}(x^2\cot\alpha_1+y^2\cot\alpha_2)}}a'(x,y,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)dtdu \\
= & \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} C_{\alpha_1}C_{\alpha_2}\overline{K_{\alpha_1,\alpha_2}(x,y,\xi,\zeta)\widehat{a}_{\alpha_1,\alpha_2}(}\xi-t,\zeta-u,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)e^{i(\xi t-t^2)cot\alpha_1+i(\zeta u-u^2)cot\alpha_2}dtdud\xi d\zeta
\end{aligned}$$

$$
\begin{aligned}
& \left|\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{K_{\alpha_1,\alpha_2}(x,y,t,u)} e^{\frac{i}{2}(x^2\cot\alpha_1+y^2\cot\alpha_2)} a'(x,y,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)dtdu \right| \\
= & \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} C_{\alpha_1}\overline{C_{\alpha_2}K_{\alpha_1,\alpha_2}(x,y,\xi,\zeta)\widehat{a}_{\alpha_1,\alpha_2}}(\xi-t,\zeta-u,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)e^{i(\xi t-t^2)cot\alpha_1+i(\zeta u-u^2)cot\alpha_2}dtdud\xi d \\
\leq & \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} |C_{\alpha_1}||\overline{C_{\alpha_2}}||\overline{K_{\alpha_1,\alpha_2}(x,y,\xi,\zeta)}||\widehat{a}_{\alpha_1,\alpha_2}(\xi-t,\zeta-u,t,u)||\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)|dtdud\xi d\zeta \\
\leq & \frac{D_l}{4\pi^2|sin\alpha_1||sin\alpha_2|}\int_{\mathbb{R}}\int_{\mathbb{R}}|\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)|\left(\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi-t|^2csc^2\alpha_1)^{-\frac{l}{2}}(1+|\zeta-u|^2csc^2\alpha_2)^{-\frac{l}{2}}d\xi d\zeta\right)dtdu \\
< & \infty \quad \text{for } l \text{ large enough.}
\end{aligned}
$$

Furthermore, we see that (21) equals

$$
\begin{aligned}
& \int_{\mathbb{R}}\int_{\mathbb{R}}\overline{C_{\alpha_1}C_{\alpha_2}}e^{-\frac{i}{2}(x^2+\xi^2)cot\alpha_1+ix\xi csc\alpha_1}e^{-\frac{i}{2}(y^2+\zeta^2)cot\alpha_2+y\zeta csc\alpha_2}\left[\overline{C_{\alpha_1}C_{\alpha_2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\widehat{a}_{\alpha_1,\alpha_2}(\xi-t,\zeta-u,t,u)\right. \\
\times & \left.\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)e^{i(\xi t-t^2)\cot\alpha_1}e^{i(\zeta u-u^2)cot\alpha_2}dtdu\right]d\xi d\zeta \\
= & \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{C_{\alpha_1}C_{\alpha_2}}e^{-\frac{i}{2}(x^2+(\xi-t)^2)cot\alpha_1+ix(\xi-t)csc\alpha_1}e^{-\frac{i}{2}(y^2+(\zeta-u)^2)cot\alpha_2+y(\zeta-u)csc\alpha_2} \\
\times & \overline{C_{\alpha_1}C_{\alpha_2}}\widehat{a}_{\alpha_1,\alpha_2}(\xi-t,\zeta-u,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)e^{i\frac{x^2}{2}\cot\alpha_1+i\frac{y^2}{2}cot\alpha_2}dtdud\xi d\zeta \\
= & \int_{\mathbb{R}}\int_{\mathbb{R}}e^{i\frac{x^2}{2}\cot\alpha_1+i\frac{y^2}{2}cot\alpha_2}\overline{K_{\alpha_1,\alpha_2}(x,y,t,u)}a'(x,y,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)dtdu.
\end{aligned}
$$

This will prove Theorem 5. □

Theorem 6. *Let $a(x,y,\xi,\zeta)$ be a symbol, and $\overline{a(x,y,\xi,\zeta)}$ its complex conjugate, the operator $A(x,y,D'_{x,y})$ associated to $a(x,y,\xi,\zeta)$, operator $\overline{\mathscr{A}(x,y,D'_{x,y})}$ associated to $\overline{a(x,y,\xi,\zeta)}$. Then, we have the equality*

$$
\langle A(x,y,D'_{x,y})\psi,\varphi\rangle_{L^2(\mathbb{R}\times\mathbb{R})} = \langle\psi,\overline{\mathscr{A}(x,y,D'_{x,y})}\varphi\rangle_{L^2(\mathbb{R}\times\mathbb{R})} \qquad \forall\psi,\ \varphi\in\mathscr{S}(\mathbb{R}\times\mathbb{R}).
$$

Proof. It will be sufficient to show that for $\psi,\ \varphi\in\mathscr{S}(\mathbb{R}\times\mathbb{R})$. We have first of all

$$
[\overline{\mathscr{A}(x,y,D'_{x,y})\varphi}](x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}}\overline{K_{\alpha_1,\alpha_2}(x,y,\xi,\zeta)}\ \overline{a(x,y,\xi,\zeta)}\widehat{\varphi}_{\alpha_1,\alpha_2}(\xi,\zeta)d\xi d\zeta,\ \forall\varphi\in\mathscr{S}(\mathbb{R}\times\mathbb{R})\ (\text{ Theorem5 }).
$$

Hence, we get, when

$$
\langle\psi,\varphi\rangle_{L^2(\mathbb{R}\times\mathbb{R})} = \int_{\mathbb{R}}\int_{\mathbb{R}}\psi(\xi)\overline{\varphi(\zeta)}d\xi d\zeta,
$$

the equality

$$
\begin{aligned}
\langle\psi,\overline{\mathscr{A}(x,y,D'_{x,y})}\varphi\rangle_{L^2(\mathbb{R}\times\mathbb{R})} &= \int_{\mathbb{R}}\int_{\mathbb{R}}\psi(x,y)\overline{\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{K_{\alpha_1,\alpha_2}(x,y,\xi,\zeta)}\ \overline{a(x,y,\xi,\zeta)}\widehat{\varphi}_{\alpha_1,\alpha_2}(\xi,\zeta)d\xi d\zeta\right)}dxdy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\psi(x,y)K_{\alpha_1,\alpha_2}(x,y,\xi,\zeta)a(x,y,\xi,\zeta)\overline{\widehat{\varphi}_{\alpha_1,\alpha_2}(\xi,\zeta)}d\xi d\zeta dxdy. \quad (22)
\end{aligned}
$$

Now by Parsevel's formula, we have obtain, using also Theorem 3

$$\begin{aligned}\langle A(x,y,D'_{x,y})\psi,\varphi\rangle_{L^2(\mathbb{R}\times\mathbb{R})} &= \langle \mathscr{F}_{\alpha_1,\alpha_2}[A(x,y,D'_{x,y})\psi],\widehat{\varphi}_{\alpha_1,\alpha_2}\rangle_{L^2(\mathbb{R}\times\mathbb{R})}\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\mathscr{F}_{\alpha_1,\alpha_2}[A(x,y,D'_{x,y})\psi](\xi,\zeta)\overline{\widehat{\varphi}_{\alpha_1,\alpha_2}(\xi,\zeta)}d\xi d\zeta\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\alpha_1,\alpha_2}(x,y,\xi,\zeta)a(x,y,\xi,\zeta)\psi(x,y)\overline{\widehat{\varphi}_{\alpha_1,\alpha_2}(\xi,\zeta)}dxdyd\xi d\zeta \qquad (23)\end{aligned}$$

From (22) and (23), we obtain the equality

$$\langle A(x,y,D'_{x,y})\psi,\varphi\rangle_{L^2(\mathbb{R}\times\mathbb{R})} = \langle \psi,\overline{\mathscr{A}(x,y,D'_{x,y})}\varphi\rangle_{L^2(\mathbb{R}\times\mathbb{R})} \qquad \forall\psi,\ \varphi\in\mathscr{S}(\mathbb{R}\times\mathbb{R}).$$

This will prove Theorem 6. □

Theorem 7. *We have the relation*

$$\|\left[A(x,y,D'_{x,y})-\mathscr{A}(x,y,D'_{x,y})\right]\phi\|_s \le C'_{\alpha_1,\alpha_2}\|\phi\|_s, \quad \forall s\in\mathbb{R}, \quad \forall\phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R}).$$

Proof. It is known that $A(x,y,D'_{x,y})\phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R})$ and that we have

$$\begin{aligned}\left[\mathscr{F}_{\alpha_1,\alpha_2}\big(A(x,y,D'_{x,y})\big)\phi\right](t,u) &= a(\infty,\infty,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)+\overline{C_{\alpha_1}}\,\overline{C_{\alpha_2}}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{i(t\lambda_1-\lambda_1^2)cot\alpha_1+i(u\lambda_2-\lambda_2^2)cot\alpha_2}\\ &\quad\times\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2\end{aligned}$$

(Coupled fractional Fourier transform in$\mathscr{S}'(\mathbb{R}\times\mathbb{R})$). The same is valid for $\mathscr{A}(x,y,D'_{x,y})$ and

$$\begin{aligned}\left[\mathscr{F}_{\alpha_1,\alpha_2}\big(\mathscr{A}(x,y,D'_{x,y})\big)\phi\right](t,u) &= a(\infty,\infty,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)+\overline{C_{\alpha_1}}\,\overline{C_{\alpha_2}}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{i(t\lambda_1-\lambda_1^2)cot\alpha_1+i(u\lambda_2-\lambda_2^2)cot\alpha_2}\\ &\quad\times\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,\lambda_1,\lambda_2)\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2.\end{aligned}$$

Hence, we obtain, with Coupled fractional Fourier transform in $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$

$$\begin{aligned}&\left[\mathscr{F}_{\alpha_1,\alpha_2}\big(A(x,y,D'_{x,y})-\mathscr{A}(x,y,D'_{x,y})\big)\phi\right](t,u)\\ =\ &\overline{C_{\alpha_1}}\,\overline{C_{\alpha_2}}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{i(t\lambda_1-\lambda_1^2)cot\alpha_1+i(u\lambda_2-\lambda_2^2)cot\alpha_2}\left[\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)-\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,\lambda_1,\lambda_2)\right]\\ \times\ &\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2.\end{aligned}$$

Therefore, we will have to estimate the norm $L_2(\mathbb{R}\times\mathbb{R})$ of the expression

$$\begin{aligned}
\mathscr{U}_s(t,u) &= (1+|t|^2)^{\frac{s}{2}}(1+|u|^2)^{\frac{s}{2}}\overline{C_{\alpha_1}}\,\overline{C_{\alpha_2}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda_1^2)cot\alpha_1+i(u\lambda_2-\lambda_2^2)cot\alpha_2} \\
&\quad\times\left[\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)-\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,\lambda_1,\lambda_2)\right]\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 \\
&= \overline{C_{\alpha_1}}\,\overline{C_{\alpha_2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(1+|t|^2)^{\frac{s}{2}}}{(1+|\lambda_1|^2)^{\frac{s}{2}}}\frac{(1+|u|^2)^{\frac{s}{2}}}{(1+|\lambda_2|^2)^{\frac{s}{2}}}e^{i(t\lambda_1-\lambda_1^2)cot\alpha_1+i(u\lambda_2-\lambda_2^2)cot\alpha_2} \\
&\quad\times\left[\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)-\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,\lambda_1,\lambda_2)\right] \\
&\quad\times(1+|\lambda_1|^2)^{\frac{s}{2}}(1+|\lambda_2|^2)^{\frac{s}{2}}\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 \\
|\mathscr{U}_s(t,u)| &\leq |\overline{C_{\alpha_1}}\,\overline{C_{\alpha_2}}|2^{|s|}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|t-\lambda_1|^2)^{\frac{|s|}{2}}(1+|u-\lambda_2|^2)^{\frac{|s|}{2}}\Big[|\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)| \\
&\quad+|\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,\lambda_1,\lambda_2)|\Big](1+|\lambda_1|^2)^{\frac{s}{2}}(1+|\lambda_2|^2)^{\frac{s}{2}}|\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)|d\lambda_1 d\lambda_2 \\
&\leq |\overline{C_{\alpha_1}}\,\overline{C_{\alpha_2}}|2^{|s|}D_l\int_{\mathbb{R}}\int_{\mathbb{R}}2(1+|t-\lambda_1|^2)^{\frac{|s|}{2}}(1+|u-\lambda_2|^2)^{\frac{|s|}{2}}(1+|t-\lambda_1|^2\csc\alpha_1)^{-\frac{l}{2}} \\
&\quad\times(1+|u-\lambda_2|^2\csc\alpha_2)^{-\frac{l}{2}}(1+|\lambda_1|^2)^{\frac{s}{2}}(1+|\lambda_2|^2)^{\frac{s}{2}}\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}h_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2)f_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 \quad \text{(say)} \\
&= (h_{\alpha_1,\alpha_2}*f_{\alpha_1,\alpha_2})(t,u).
\end{aligned}$$

If l is large, $h_{\alpha_1,\alpha_2}\in L_1(\mathbb{R}\times\mathbb{R})$. Also, since $\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)\in\mathscr{S}(\mathbb{R}\times\mathbb{R})$, $f_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)\in L_2(\mathbb{R}\times\mathbb{R})$.
Then $(h_{\alpha_1,\alpha_2}*f_{\alpha_1,\alpha_2})(t,u)$ belongs to $L_2(\mathbb{R}\times\mathbb{R})$ and the inequality

$$||h_{\alpha_1,\alpha_2}*f_{\alpha_1,\alpha_2}||\leq||h_{\alpha_1,\alpha_2}||_{L_1(\mathbb{R}\times\mathbb{R})}||f_{\alpha_1,\alpha_2}||_{L_2(\mathbb{R}\times\mathbb{R})}.$$

It implies that

$$||\mathscr{U}_s||_{L_2(\mathbb{R}\times\mathbb{R})}\leq C'_{\alpha_1,\alpha_2}||\phi||_s.$$

Hence

$$||\left[A(x,y,D'_{x,y})-\mathscr{A}(x,y,D'_{x,y})\right]\phi||_s\leq C'_{\alpha_1,\alpha_2}||\phi||_s,\quad\forall s\in\mathbb{R},\quad\forall\phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R}).$$

This will prove Theorem 7. □

References

[1] Victor Namias. The fractional order fourier transform and its application to quantum mechanics. *IMA Journal of Applied Mathematics*, 25(3):241–265, 1980.

[2] AC McBride and FH Kerr. On namias's fractional fourier transforms. *IMA Journal of applied mathematics*, 39(2):159–175, 1987.

[3] Fiona H Kerr. Namias' fractional fourier transforms on l2 and applications to differential equations. *Journal of mathematical analysis and applications*, 136(2):404–418, 1988.

[4] T Alieva, Vicente Lopez, Fernando Agulló-López, and LB Almeida. The fractional fourier transform in optical propagation problems. *Journal of modern optics*, 41(5):1037–1044, 1994.

[5] Luis B Almeida. The fractional fourier transform and time-frequency representations. *IEEE Transactions on signal processing*, 42(11):3084–3091, 1994.

[6] Luis B Almeida. Product and convolution theorems for the fractional fourier transform. *IEEE Signal Processing Letters*, 4(1):15–17, 1997.

[7] Adolf W Lohmann and Bernard H Soffer. Relationships between the radon–wigner and fractional fourier transforms. *JOSA A*, 11(6):1798–1801, 1994.

[8] Haldun M Ozaktas and David Mendlovic. Fourier transforms of fractional order and their optical interpretation. *Optics Communications*, 101(3-4):163–169, 1993.

[9] Ahmed I Zayed. A convolution and product theorem for the fractional fourier transform. *IEEE Signal processing letters*, 5(4):101–103, 1998.

[10] Ahmed I. Zayed. Fractional fourier transform of generalized functions. *Integral Transforms and Special Functions*, 7(3-4):299–312, 1998.

[11] Ahmed I Zayed. On the relationship between the fourier and fractional fourier transforms. *IEEE signal processing letters*, 3(12):310–311, 1996.

[12] Hongzhe Dai, Zhibao Zheng, and Wei Wang. A new fractional wavelet transform. *Communications in Nonlinear Science and Numerical Simulation*, 44:19–36, 2017.

[13] Akhilesh Prasad, Santanu Manna, Ashutosh Mahato, and VK Singh. The generalized continuous wavelet transform associated with the fractional fourier transform. *Journal of computational and applied mathematics*, 259:660–671, 2014.

[14] Akhilesh Prasad and Praveen Kumar. Pseudo-differential operator associated with the fractional fourier transform. *Mathematical Communications*, 21(1):115–126, 2016.

[15] Yuri Luchko. Some schemata for applications of the integral transforms of mathematical physics. *Mathematics*, 7(3):254, 2019.

[16] RS Pathak, Akhilesh Prasad, and Manish Kumar. Fractional fourier transform of tempered distributions and generalized pseudo-differential operator. *Journal of Pseudo-Differential Operators and Applications*, 3:239–254, 2012.

[17] Haldun M Ozaktas and M Alper Kutay. The fractional fourier transform. In *2001 European Control Conference (ECC)*, pages 1477–1483. IEEE, 2001.

[18] Ahmed I Zayed. A class of fractional integral transforms: a generalization of the fractional fourier transform. *IEEE transactions on signal processing*, 50(3):619–627, 2002.

[19] Akhilesh Prasad and Vishal Kumar Singh. On pseudo-differential operator associated with bessel operator. *Int. J. Contemp. Math. Sciences*, 6(25):1237–1243, 2011.

[20] Akhilesh Prasad and Kanailal Mahato. On the sobolev boundedness results of the product of pseudo-differential operators involving a couple of fractional hankel transforms. *Acta Mathematica Sinica, English Series*, 34(2):221–232, 2018.

[21] Abhisekh Shekhar and Nawin Kumar Agrawal. Fractional fourier transform on sobolev spaces connected to negative definite functions. *Asian Journal of Pure and Applied Mathematics*, pages 112–122, 2023.

[22] Abhisekh Shekhar. Fractional fourier-bessel type transform. In *AIP Conference Proceedings*, volume 3180. AIP Publishing, 2024.

[23] Abhisekh Shekhar and Nawin Kumar Agrawal. Inequality and estimate of generalized pseudo-differential operators involving fractional fourier transform. In *American Institute of Physics Conference Series*, volume 3087, page 090001, 2024.

[24] Abhisekh Shekhar and Nawin Agrawal. Generalized pseudo-differential operators associated with symbol classes involving fractional fourier transform. *Serdica Mathematical Journal*, 48(4):247–270, 2022.

[25] Shraban Das, Kanailal Mahato, and Ahmed I Zayed. Characterization of pseudo-differential operators associated with the coupled fractional fourier transform. *Axioms*, 13(5):296, 2024.

[26] Ahmed Zayed. Two-dimensional fractional fourier transform and some of its properties. *Integral Transforms and Special Functions*, 29(7):553–570, 2018.

[27] R Kamalakkannan, R Roopkumar, and A Zayed. On the extension of the coupled fractional fourier transform and its properties. *Integral Transforms and Special Functions*, 33(1):65–80, 2022.

[28] S Zaidman. Pseudo-differential operators. *Annali di Matematica Pura ed Applicata*, 92(1):345–399, 1972.

[29] Jean-Marc Bouclet. An introduction to pseudo-differential operators. *Lecture Notes, Available at http://www. math. univ-toulouse. fr/˜ bouclet*, 2012.

[30] Man-Wah Wong. *Introduction To Pseudo-differential Operators, An*, volume 6. World Scientific Publishing Company, 2014.

[31] Akhilesh Prasad and Manish Kumar. Product of two generalized pseudo-differential operators involving fractional fourier transform. *Journal of Pseudo-Differential Operators and Applications*, 2(3):355–365, 2011.

[32] RS Pathak, Akhilesh Prasad, and Manish Kumar. An n-dimensional pseudo-differential operator involving the hankel transformation. *Proceedings-Mathematical Sciences*, 122(1):99–120, 2012.

[33] Joseph J Kohn and Louis Nirenberg. An algebra of pseudo-differential operators. *Communications on Pure and Applied Mathematics*, 18(1-2):269–305, 1965.

[34] Samuel Zaidman. *Distributions and pseudo-differential operators*, volume 248. Harlow, England: Longman Scientific & Technical, 1991.

[35] Ram S Pathak and Pradip K Pandey. A class of pseudo-differential operators associated with bessel operators. *Journal of mathematical analysis and applications*, 196(2):736–747, 1995.

[36] RS Pathak and S Pathak. Certain pseudo-differential operator associated with the bessel operator. *INDIAN JOURNAL OF PURE & APPLIED MATHEMATICS*, 31(3):309–317, 2000.

[37] Ram Shankar Pathak and S Pathak. The pseudodifferential operator a (x, d). *International Journal of Mathematics and Mathematical Sciences*, 2004(8):407–419, 2004.

[38] Abdessalem Gasmi and Anis El Garna. Properties of the linear multiplier operator for the weinstein transform and applications. *Electronic Journal of Differential Equations*, 124:1–18, 2017.

[39] Miloud Assal. Pseudo-differential operators associated with laguerre hypergroups. *Journal of computational and applied mathematics*, 233(3):617–620, 2009.

[40] N Ben Salem and A Dachraoui. Pseudo-differential operators associated with the jacobi differential operator. *Journal of mathematical analysis and applications*, 220(1):365–381, 1998.