Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 8s (2025)

A Numerical Approach for Solving Fuzzy Differential Equation Using Enhanced Euler's Methods Based on Contra Harmonic Mean and Centroidal Mean

A. Hari Ganesh[1*], R. Diwahar[2]

[1*]Assistant Professor and [2]Research Scholar (Full Time), PG & Research Department of Mathematic, Poompuhar College (Autonomous), (Affiliated to Bharathidasan University), Melaiyur – 609 107, Mayiladuthurai (Dt.), Tamil Nadu, India.

Article History:

Received: *24-10-2024*

Revised:*25-11-2024*

Accepted:*27-12-2024*

Abstract:

Objectives: Fuzzy differential equations (FDEs) are crucial for modeling dynamic systems in science, economics, and engineering due to their unpredictable nature and the need for numerical methods for precise solutions. The objective of this work is to develop improved Euler's techniques for numerically solving fuzzy differential equations (FDEs) in order to produce approximate solutions for problems that are too complicated to be stated exactly.

Methods: This study proposes improved Euler's approaches for solving differential equations with fuzzy initial conditions, including Euler's Method, Modified Euler's Method, and Improved Euler's Method employing Contra Harmonic Mean and Centroidal Mean. Based on Zadeh's extension concept for fuzzy sets, we extend Euler's classical methods to address this dependence problem in a fuzzy setting.

Findings: A numerical example that contrasts the suggested approaches with the traditional Euler's Method is provided. The efficiency of the suggested methods is shown by numerical solutions and their geometrical representations. Furthermore, the numerical example demonstrates the acceptable accuracy of the improved Euler's techniques.

Novelty: In this research proposal, we sought to solve first-order differential equations utilizing two creative approaches: the Contra Harmonic Mean and the Centroidal Mean of enhanced Euler's approaches for fuzzy primary value. The suggested methods yield great results and are quick, accurate, and easy to use.

Keywords: Fuzzy Initial Value Problem, Euler's Method, Modified Euler's Method, Improved Euler's Method, Contra Harmonic Mean, Centroidal Mean.

1. Introduction:

Since most differential equations cannot be solved directly and must instead be approximated by approximation, numerical methods are crucial. Of all the numerical techniques, Euler's Method is the most straightforward. Its foundation is using a series of successively calculated tangent line approximations to approximate the graph of a solution, y(x). Furthermore, Euler's method is the foundation for the majority of other approaches. These days, researchers are interested in learning how to use the fuzzy version of the Euler method to solve fuzzy differential equations **[1, 2]**. A modified Euler technique for solving fuzzy differential equations was described in **[3]**. The graphical comparison of the fourth order RK technique, modified Euler's method and Euler's method on numerical solutions of the first order initial value problem was discussed in **[4]** and **[5]**. In order to handle fuzzy initial value problems without converting fuzzy differential equations into a system of crisp differential equations, Euler's fuzzy form was devised in **[6]**. A numerical procedure based on a modified Euler's method was used to determine the numerical solution of a fuzzy ordinary differential equation in **[7]**. The codomain of the membership function of the fuzzy topological space was extended from [0, 1] to the unit disk in the complex plane in order to achieve the numerical solution of complex fuzzy differential equations using Euler and Taylor techniques in **[8]**. By applying the harmonic mean and cubic mean of Euler's modified technique, the first order differential equations were solved in **[9]**. A modified Euler method was introduced for solving fuzzy differential equations under generalized differentiability in **[10, 11]**. It also estimated the differential equations by using two stages Predictor – Corrector algorithm with local truncation error of order two. Two standard numerical methods such as Euler and Runge – Kutta fourth order methods were presented to solve second order initial value problem for ordinary differential equation in **[12, 13]**. In order to improve the accuracy of numerical solution of fuzzy differential equations, we shall enhance Euler's Method, Modified Euler's Method, and Improved Euler's Method based on Contra Harmonic Mean and Centroidal Mean in this paper.

The framework of this paper is as follows: We review some basic ideas about fuzzy sets and fuzzy integers in Section 2. In Section 3, we improve on the conventional Euler's Methods, which are based on the Centroidal Mean and Contra Harmonic Mean, to find the numerical solutions of fuzzy differential equations with lower error values. Section 4 presents the fuzzy initial value problem and the fuzzy version of the suggested techniques. To demonstrate the efficacy of our approach in comparison to the conventional Euler's Method, we use it in section 5 to obtain numerical solutions for the fuzzy initial values problem. Section 6 offers a succinct conclusion.

2. Prologues:

Some helpful definitions of fuzzy sets and fuzzy numbers that are required or established in this paper will be introduced in this section.

Definition

In the universal set X, a fuzzy set $\widetilde{A}$ is defined as $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)); x \in X, \mu_{\widetilde{A}}(x) \in (0,1]\}$. In this case, the membership function's grade is $\mu_{\widetilde{A}}(x) \to [0,1]$, and the fuzzy set $\widetilde{A}$'s grade value of $x \in X$ is $\mu_{\widetilde{A}}(x)$.

Definition

The crisp set $\tilde{A}^s$ that includes every element of the universal set X whose membership grades in $\widetilde{A}$ are higher than or equal to the designated value of α is the α -cut of a fuzzy set $\widetilde{A}$. It is indicated by

$$\tilde{A}^s = \{x \in X \mid \mu_{\widetilde{A}}(x) \geq s\}$$

Where $0 \leq s \leq 1$.

Definition

The membership function of a fuzzy number $\widetilde{A}$, which is a subset of real line R, has the following characteristics:

1. In its domain, $\mu_{\widetilde{A}}(x)$is piecewise continuous.
2. $\widetilde{A}$ is normal, meaning that a $x_0 \in X$ such that $\mu_{\widetilde{A}}(x_0) = 1.$
3. $\widetilde{A}$ is convex; that is, $\mu_{\widetilde{A}}(\lambda\, x_1 + (1-\lambda)\, x_2) \geq min\big(\mu_{\widetilde{A}}(\,x_1), \mu_{\widetilde{A}}(\,x_2)\big)$ and $\forall\, x_1, x_2 \in X$.

Definition

The membership function of a triangular fuzzy number $\tilde{u}$ is described as follows. It may be expressed by a triplet (ρ_1,ρ_2,ρ_3).

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \begin{cases} \mathbf{0}, & if\ x < \rho_1 \\ \dfrac{\boldsymbol{x}-\rho_1}{\rho_2-\rho_1}, & if\ \rho_1 \le x \le \rho_2 \\ \dfrac{\rho_3 - x}{\rho_3-\rho_2}, & if\ \rho_2 \le x \le \rho_3 \\ \mathbf{0}, & if\ x > \rho_3 \end{cases}$$

The s - level of the fuzzy number $\tilde{\boldsymbol{u}}$ is $\tilde{\boldsymbol{u}}_s = [\rho_1 + (\rho_2 - \rho_3)s, \rho_3 - (\rho_3 - \rho_2)s]$ for any $s \in [0,1]$.

3. Methodology

This section will cover the completion of the Euler, Modified Euler and Improved Euler approaches based on Contra Harmonic Mean and Centroidal Mean for solving the first order differential equations with fuzzy initial conditions.

3.1. Enhanced Euler's Methods based on Contra Harmonic Mean and Centroidal Mean:

3.1.1 Euler's Method based on Contra Harmonic Mean

Take note of a, b, and c as the Contra Harmonic Sequence's components. The formula for calculating the Contra Harmonic mean is $c = \frac{a^2+b^2}{a+b}$. Next, the two center points expected Contra Harmonic mean is determined by

$$\left(\frac{t_0^2+t_1^2}{t_0+t_1}, \frac{u_0^2+u_1^2}{u_0+u_1}\right).$$

The formula may be written as

$$\left(\frac{t_0^2+(t_0+h)^2}{t_0+(t_0+h)}, \frac{u_0^2+\left(u_0+hf(t_0,u_0)\right)^2}{u_0+\left(u_0+hf(t_0,u_0)\right)}\right).$$

We may create a new equation denoted as, by using an equation in the vicinity of a point, such as $P(t_0,u_0)$, and the gradient up the equation.

$$u_1 = u_0 + hf\left(\frac{t_0^2+(t_0+h)^2}{t_0+(t_0+h)}, \frac{u_0^2+\left(u_0+hf(t_0,u_0)\right)^2}{u_0+\left(u_0+hf(t_0,u_0)\right)}\right).$$

The Euler approach will be more stable if it estimates u_{n+1}using the modified and stable slope function at the predicted center points of (t_0,u_0) and (t_1,u_1). The equation stated above is known as Euler's Modified Contra Harmonic Mean. It is characterized by

$$u_{n+1} = u_n + hf\left(\frac{t_n^2+(t_n+h)^2}{t_n+(t_n+h)}, \frac{u_n^2+\left(u_n+hf(t_n,u_n)\right)^2}{u_n+\left(u_n+hf(t_n,u_n)\right)}\right).$$

3.1.2 Modified Euler's Method based on Contra Harmonic Mean

Take note of a, b, and c as the Contra Harmonic Sequence's components. The formula for calculating the Contra Harmonic mean is $c = \frac{a^2+b^2}{a+b}$. Next, the two center points expected Contra Harmonic mean is determined by

$$\left(\frac{t_0^2+t_1^2}{t_0+t_1}, \frac{u_0^2+u_1^2}{u_0+u_1}\right).$$

The formula may be written as

$$\left(\frac{t_0^2+(t_0+h)^2}{t_0+(t_0+h)},\frac{u_0^2+\left(u_0+hf\left(t_0+\frac{h}{2},u_0+\frac{h}{2}f(t_0,u_0)\right)\right)^2}{u_0+\left(u_0+hf\left(t_0+\frac{h}{2},u_0+\frac{h}{2}f(t_0,u_0)\right)\right)}\right).$$

We may create a new equation denoted as, by using an equation in the vicinity of a point, such as $P(t_0,u_0)$, and the gradient up the equation.

$$u_1=u_0+hf\left(\begin{array}{c}t_0+\frac{h}{2},\\ u_0+\frac{h}{2}f\left(\frac{t_0^2+(t_0+h)^2}{t_0+(t_0+h)},\frac{u_0^2+\left(u_0+hf\left(t_0+\frac{h}{2},u_0+\frac{h}{2}f(t_0,u_0)\right)\right)^2}{u_0+\left(u_0+hf\left(t_0+\frac{h}{2},u_0+\frac{h}{2}f(t_0,u_0)\right)\right)}\right)\end{array}\right).$$

The Euler approach will be more stable if it estimates u_{n+1}using the modified and stable slope function at the predicted center points of (t_0,u_0) and (t_1,u_1). The equation stated above is known as Euler's Modified Contra Harmonic Mean. It is characterized by

$$u_{n+1}=u_n+hf\left(\begin{array}{c}t_n+\frac{h}{2},\\ u_n+\frac{h}{2}f\left(\frac{t_n^2+(t_n+h)^2}{t_n+(t_n+h)},\frac{u_n^2+\left(u_n+hf\left(t_n+\frac{h}{2},u_n+\frac{h}{2}f(t_n,u_n)\right)\right)^2}{u_n+\left(u_n+hf\left(t_n+\frac{h}{2},u_n+\frac{h}{2}f(t_n,u_n)\right)\right)}\right)\end{array}\right).$$

3.1.3 Improved Euler's Method based on Contra Harmonic Mean

Take note of a, b, and c as the Contra Harmonic Sequence's components. The formula for calculating the Contra Harmonic mean is $c=\frac{a^2+b^2}{a+b}$. Next, the two center points expected Contra Harmonic mean is determined by

$$\left(\frac{t_0^2+t_1^2}{t_0+t_1},\frac{u_0^2+u_1^2}{u_0+u_1}\right).$$

The formula may be written as

$$\left(\frac{t_0^2+(t_0+h)^2}{t_0+(t_0+h)},\frac{u_0^2+\left(u_0+\frac{h}{2}\left(f(t_0,u_0)+f(t_0+h,u_0+hf(t_0,u_0))\right)\right)^2}{u_0+\left(u_0+\frac{h}{2}\left(f(t_0,u_0)+f(t_0+h,u_0+hf(t_0,u_0))\right)\right)}\right).$$

We may create a new equation denoted as, by using an equation in the vicinity of a point, such as $P(t_0,u_0)$, and the gradient up the equation.

$$u_1=u_0+\frac{h}{2}\left[f\left(\frac{t_0^2+(t_0+h)^2}{t_0+(t_0+h)},\frac{u_0^2+\left(u_0+\frac{h}{2}\left(f(t_0,u_0)+f(t_0+h,u_0+hf(t_0,u_0))\right)\right)^2}{u_0+\left(u_0+\frac{h}{2}\left(f(t_0,u_0)+f(t_0+h,u_0+hf(t_0,u_0))\right)\right)}\right)+\right.$$
$$\left.f\left(\begin{array}{c}t_0+h,\\ u_0+h\left(\frac{t_0^2+(t_0+h)^2}{t_0+(t_0+h)},\frac{u_0^2+\left(u_0+\frac{h}{2}\left(f(t_0,u_0)+f(t_0+h,u_0+hf(t_0,u_0))\right)\right)^2}{u_0+\left(u_0+\frac{h}{2}\left(f(t_0,u_0)+f(t_0+h,u_0+hf(t_0,u_0))\right)\right)}\right)\end{array}\right)\right].$$

The Euler approach will be more stable if it estimates u_{n+1}using the modified and stable slope function at the predicted center points of (t_0, u_0) and (t_1, u_1). The equation stated above is known as Euler's Modified Contra Harmonic Mean. It is characterized by

$$u_{n+1} = u_n + \frac{h}{2}\left[f\left(\frac{t_n^2+(t_n+h)^2}{t_n+(t_n+h)}, \frac{u_n^2+\left(u_n+\frac{h}{2}\left(f(t_n,u_n)+f\left(t_n+h,u_n+hf(t_n,u_n)\right)\right)\right)^2}{u_n+\left(u_n+\frac{h}{2}\left(f(t_n,u_n)+f\left(t_n+h,u_n+hf(t_n,u_n)\right)\right)\right)} \right) + f\left(\begin{array}{c} t_n+h, \\ u_n+h\left(\frac{t_n^2+(t_n+h)^2}{t_n+(t_n+h)}, \frac{u_n^2+\left(u_n+\frac{h}{2}\left(f(t_n,u_n)+f\left(t_n+h,u_n+hf(t_n,u_n)\right)\right)\right)^2}{u_n+\left(u_n+\frac{h}{2}\left(f(t_n,u_n)+f\left(t_n+h,u_n+hf(t_n,u_n)\right)\right)\right)} \right) \end{array} \right) \right].$$

3.1.4Euler's Method based on Centroidal Mean

Assume that the essential components of a centroidal sequence are a, b, and c. The formula to compute the Centroidal mean is $c = \frac{2(a^2+ab+b^2)}{3(a+b)}$. The expected Centroidal mean of the two midpoints is therefore given as

$$\left(\frac{2(t_0^2+t_0t_1+t_1^2)}{3(t_0+t_1)}, \frac{2(u_0^2+u_0u_1+u_1^2)}{3(u_0+u_1)} \right).$$

The mathematical equation is defined as

$$\left(\frac{2(t_0^2+t_0(t_0+h)+(t_0+h)^2)}{3(t_0+(t_0+h))}, \frac{2\left(u_0^2+u_0(u_0+hf(t_0,u_0))+(u_0+hf(t_0,u_0))^2\right)}{3\left(u_0+(u_0+hf(t_0,u_0))\right)} \right).$$

A new equation can be created and provided as follows if an equation goes through a point, such as $P(t_0, u_0)$ in the center of the slope throughout the equation.

$$u_1 = u_0 + hf\left(\begin{array}{c} \frac{2(t_0^2+t_0(t_0+h)+(t_0+h)^2)}{3(t_0+(t_0+h))}, \\ \frac{2\left(u_0^2+u_0(u_0+hf(t_0,u_0))+(u_0+hf(t_0,u_0))^2\right)}{3\left(u_0+(u_0+hf(t_0,u_0))\right)} \end{array} \right).$$

The results of the modified Euler's approach are now certain and accurate. The Centroidal Mean of the Modified Euler's Method is the name given to the preceding equation. It is possible to write it as

$$u_{n+1} = u_n + hf\left(\begin{array}{c} \frac{2(t_n^2+t_n(t_n+h)+(t_n+h)^2)}{3(t_n+(t_n+h))}, \\ \frac{2\left(u_n^2+u_n(u_n+hf(t_n,u_n))+(u_n+hf(t_n,u_n))^2\right)}{3\left(u_n+(u_n+hf(t_n,u_n))\right)} \end{array} \right).$$

3.1.5 Modified Euler's Method based on Centroidal Mean

Assume that the essential components of a centroidal sequence are a, b, and c. The formula to compute the Centroidal mean is $c = \frac{2(a^2+ab+b^2)}{3(a+b)}$. The expected Centroidal mean of the two midpoints is therefore given as

$$\left(\frac{2(t_0^2+t_0t_1+t_1^2)}{3(t_0+t_1)}, \frac{2(u_0^2+u_0u_1+u_1^2)}{3(u_0+u_1)} \right).$$

The mathematical equation is defined as

$$\left(\frac{2\left(t_0^2+t_0(t_0+h)+(t_0+h)^2\right)}{3\left(t_0+(t_0+h)\right)},\frac{2\left(u_0^2+u_0\left(u_0+hf\left(t_0+\frac{h}{2},u_0+\frac{h}{2}f(t_0,u_0)\right)\right)+\left(u_0+hf\left(t_0+\frac{h}{2},u_0+\frac{h}{2}f(t_0,u_0)\right)\right)^2\right)}{3\left(u_0+\left(u_0+hf\left(t_0+\frac{h}{2},u_0+\frac{h}{2}f(t_0,u_0)\right)\right)\right)}\right).$$

A new equation can be created and provided as follows if an equation goes through a point, such as $P(t_0,u_0)$ in the center of the slope throughout the equation.

$$u_1=u_0+hf\left(\begin{array}{c}t_0+\frac{h}{2},\\ u_0+\frac{h}{2}f\left(\begin{array}{c}\frac{2\left(t_0^2+t_0(t_0+h)+(t_0+h)^2\right)}{3\left(t_0+(t_0+h)\right)},\\ \frac{2\left(u_0^2+u_0\left(u_0+hf\left(t_0+\frac{h}{2},u_0+\frac{h}{2}f(t_0,u_0)\right)\right)+\left(u_0+hf\left(t_0+\frac{h}{2},u_0+\frac{h}{2}f(t_0,u_0)\right)\right)^2\right)}{3\left(u_0+\left(u_0+hf\left(t_0+\frac{h}{2},u_0+\frac{h}{2}f(t_0,u_0)\right)\right)\right)}\end{array}\right)\end{array}\right).$$

The results of the modified Euler's approach are now certain and accurate. The Centroidal Mean of the Modified Euler's Method is the name given to the preceding equation. It is possible to write it as

$$u_{n+1}=u_n+hf\left(\begin{array}{c}t_n+\frac{h}{2},\\ u_n+\frac{h}{2}f\left(\begin{array}{c}\frac{2\left(t_n^2+t_n(t_n+h)+(t_n+h)^2\right)}{3\left(t_n+(t_n+h)\right)},\\ \frac{2\left(u_n^2+u_n\left(u_n+hf\left(t_n+\frac{h}{2},u_n+\frac{h}{2}f(t_n,u_n)\right)\right)+\left(u_n+hf\left(t_n+\frac{h}{2},u_n+\frac{h}{2}f(t_n,u_n)\right)\right)^2\right)}{3\left(u_n+\left(u_n+hf\left(t_n+\frac{h}{2},u_n+\frac{h}{2}f(t_n,u_n)\right)\right)\right)}\end{array}\right)\end{array}\right).$$

3.1.6 Improved Euler's Method based on Centroidal Mean

Assume that the essential components of a centroidal sequence are a, b, and c. The formula to compute the Centroidal mean is $c=\frac{2(a^2+ab+b^2)}{3(a+b)}$. The expected Centroidal mean of the two midpoints is therefore given as

$$\left(\frac{2(t_0^2+t_0t_1+t_1^2)}{3(t_0+t_1)},\frac{2(u_0^2+u_0u_1+u_1^2)}{3(u_0+u_1)}\right).$$

The mathematical equation is defined as

$$\left(\begin{array}{c}\frac{2\left(t_0^2+t_0(t_0+h)+(t_0+h)^2\right)}{3\left(t_0+(t_0+h)\right)},\\ \frac{2\left(u_0^2+u_0\left(u_0+\frac{h}{2}\left(f(t_0,u_0)+f(t_0+h,u_0+hf(t_0,u_0))\right)\right)+\left(u_0+\frac{h}{2}\left(f(t_0,u_0)+f(t_0+h,u_0+hf(t_0,u_0))\right)\right)^2\right)}{3\left(u_0+\left(u_0+\frac{h}{2}\left(f(t_0,u_0)+f(t_0+h,u_0+hf(t_0,u_0))\right)\right)\right)}\end{array}\right).$$

A new equation can be created and provided as follows if an equation goes through a point, such as $P(t_0,u_0)$ in the center of the slope throughout the equation.

$$u_1 = u_0 + \frac{h}{2}\left[f\left(\frac{2\left(t_0^2 + t_0(t_0+h) + (t_0+h)^2\right)}{3\left(t_0+(t_0+h)\right)}, \frac{2\left(u_0^2 + u_0\left(u_0+\frac{h}{2}\left(f(t_0,u_0)+f(t_0+h,u_0+hf(t_0,u_0))\right)\right) + \left(u_0+\frac{h}{2}\left(f(t_0,u_0)+f(t_0+h,u_0+hf(t_0,u_0))\right)\right)^2\right)}{3\left(u_0+\left(u_0+\frac{h}{2}\left(f(t_0,u_0)+f(t_0+h,u_0+hf(t_0,u_0))\right)\right)\right)} \right) + f\left(t_0+h, u_0 + h\left(\frac{2\left(t_0^2 + t_0(t_0+h) + (t_0+h)^2\right)}{3\left(t_0+(t_0+h)\right)}, \frac{2\left(u_0^2 + u_0\left(u_0+\frac{h}{2}\left(f(t_0,u_0)+f(t_0+h,u_0+hf(t_0,u_0))\right)\right) + \left(u_0+\frac{h}{2}\left(f(t_0,u_0)+f(t_0+h,u_0+hf(t_0,u_0))\right)\right)^2\right)}{3\left(u_0+\left(u_0+\frac{h}{2}\left(f(t_0,u_0)+f(t_0+h,u_0+hf(t_0,u_0))\right)\right)\right)} \right)\right)\right].$$

The results of the modified Euler's approach are now certain and accurate. The Centroidal Mean of the Modified Euler's Method is the name given to the preceding equation. It is possible to write it as

$$u_{n+1} = u_n + \frac{h}{2}\left[f\left(\frac{2\left(t_n^2 + t_n(t_n+h) + (t_n+h)^2\right)}{3\left(t_n+(t_n+h)\right)}, \frac{2\left(u_n^2 + u_n\left(u_n+\frac{h}{2}\left(f(t_n,u_n)+f(t_n+h,u_n+hf(t_n,u_n))\right)\right) + \left(u_n+\frac{h}{2}\left(f(t_n,u_n)+f(t_n+h,u_n+hf(t_n,u_n))\right)\right)^2\right)}{3\left(u_n+\left(u_n+\frac{h}{2}\left(f(t_n,u_n)+f(t_n+h,u_n+hf(t_n,u_n))\right)\right)\right)} \right) + f\left(t_n+h, u_n + h\left(\frac{2\left(t_n^2 + t_n(t_n+h) + (t_n+h)^2\right)}{3\left(t_n+(t_n+h)\right)}, \frac{2\left(u_n^2 + u_n\left(u_n+\frac{h}{2}\left(f(t_n,u_n)+f(t_n+h,u_n+hf(t_n,u_n))\right)\right) + \left(u_n+\frac{h}{2}\left(f(t_n,u_n)+f(t_n+h,u_n+hf(t_n,u_n))\right)\right)^2\right)}{3\left(u_n+\left(u_n+\frac{h}{2}\left(f(t_n,u_n)+f(t_n+h,u_n+hf(t_n,u_n))\right)\right)\right)} \right)\right)\right].$$

3.2. Fuzzy Initial Value Problem

3.2.1. Technique for precisely solving FIVP

Look over the FIVP

$$u'(t) = \begin{cases} f(t,u) \\ u(t_0) = \left(\underline{u}_0, u_0^m, \overline{u}_0\right) \end{cases}$$

In this case, a fuzzy initial condition is expressed in terms of fuzzy triangles. The FIVP’s solution is found by using fuzzy Initial conditions.

3.2.2. Proposed Methods to FIVP

Here, we investigate the fuzzy initial value problem as

$$u'(t) = \begin{cases} f(t,u) \\ u(t_0) = \left(\underline{u}_0, u_0^m, \overline{u}_0\right) \end{cases}$$

Using an s-cut method, the triangular fuzzy initial condition may be represented as

$$\left[\left(u^m - \underline{u}_0\right)s + \underline{u}_0, \overline{u}_0 + \left(u^m - \overline{u}_0\right)s\right], 0 \leq s \leq 1.$$

The FIVP has been intended to be solved using the enhanced Euler's approaches. Now, the enhanced Euler's approaches are used to assess all new upper and lower bound feasible permutations. The results of calculating the grid points as t are displayed below.

$$\underline{u}_{n+1}^{(1)}(t_{n+1}:s) = \underline{u}(t_n:s) + F[u(t_n:s)],$$
$$\overline{u}_{n+1}^{(1)}(t_{n+1}:s) = \overline{u}(t_n:s) + G[u(t_n:s)],$$
$$\underline{u}_{n+1}^{(2)}(t_{n+1}:s) = \underline{u}(t_n:s) + G[u(t_n:s)],$$
$$\overline{u}_{n+1}^{(2)}(t_{n+1}:s) = \overline{u}(t_n:s) + F[u(t_n:s)],$$

Then, we take the values from lowest to maximum, which are meant for the inferior and superior values of the variable u in order for it to yield more precise and superior answers.

$$\underline{u}_{n+1} = min\{\underline{u}(t_n:s) + F[u(t_n:s)], \underline{u}(t_n:s) + G[u(t_n:s)]\},$$
$$\overline{u}_{n+1} = max\{\overline{u}(t_n:s) + G[u(t_n:s)], \overline{u}(t_n:s) + F[u(t_n:s)]\}.$$

3.2.3. Enhanced Euler's Methods based on Contra Harmonic Mean and Centroidal Mean to FIVP

3.2.3.1. Euler's Method based on Contra Harmonic Mean to FIVP

The Modified Euler's method's expected Contra Harmonic mean is precisely defined by

$$\underline{u}_{n+1} = \underline{u}(t_n:s) + F\big(u(t_n:s)\big),$$
$$\overline{u}_{n+1} = \overline{u}(t_n:s) + G\big(u(t_n:s)\big).$$

Where

$$F = hf\left(\frac{(t_n:s)^2 + ((t_n:s)+h)^2}{(t_n:s) + ((t_n:s)+h)}, \frac{(\underline{u}_n:s)^2 + \left((\underline{u}_n:s) + hf\left((t_n:s),(\underline{u}_n:s)\right)\right)^2}{(\underline{u}_n:s) + \left((\underline{u}_n:s) + hf\left((t_n:s),(\underline{u}_n:s)\right)\right)}\right),$$

$$G = hf\left(\frac{(t_n:s)^2 + ((t_n:s)+h)^2}{(t_n:s) + ((t_n:s)+h)}, \frac{(\overline{u}_n:s)^2 + \left((\overline{u}_n:s) + hf\left((t_n:s),(\overline{u}_n:s)\right)\right)^2}{(\overline{u}_n:s) + \left((\overline{u}_n:s) + hf\left((t_n:s),(\overline{u}_n:s)\right)\right)}\right).$$

3.2.3.2. Modified Euler's Method based on Contra Harmonic Mean to FIVP

The Modified Euler's method's expected Contra Harmonic mean is precisely defined by

$$\underline{u}_{n+1} = \underline{u}(t_n:s) + F\big(u(t_n:s)\big),$$
$$\overline{u}_{n+1} = \overline{u}(t_n:s) + G\big(u(t_n:s)\big).$$

Where

$$F = hf\left(\begin{array}{c}(t_n:s) + \frac{h}{2}, \\ (\underline{u}_n:s) + \frac{h}{2} f\left(\frac{(t_n:s)^2 + ((t_n:s)+h)^2}{(t_n:s) + ((t_n:s)+h)}, \frac{(\underline{u}_n:s)^2 + \left((\underline{u}_n:s) + hf\left((t_n:s)+\frac{h}{2}, (\underline{u}_n:s) + \frac{h}{2} f\left((t_n:s),(\underline{u}_n:s)\right)\right)\right)^2}{(\underline{u}_n:s) + \left((\underline{u}_n:s) + hf\left((t_n:s)+\frac{h}{2}, (\underline{u}_n:s) + \frac{h}{2} f\left((t_n:s),(\underline{u}_n:s)\right)\right)\right)}\right)\end{array}\right),$$

$$G = hf\left(\begin{array}{c} (t_n : s) + \frac{h}{2}, \\ (\overline{u}_n : s) + \frac{h}{2} f\left(\frac{(t_n:s)^2 + \left((t_n:s)+h\right)^2}{(t_n:s) + \left((t_n:s)+h\right)}, \frac{(\overline{u}_n:s)^2 + \left((\overline{u}_n:s) + hf\left((t_n:s)+\frac{h}{2}, (\overline{u}_n:s) + \frac{h}{2} f\left((t_n:s),(\overline{u}_n:s)\right)\right)\right)^2}{(\overline{u}_n:s) + \left((\overline{u}_n:s) + hf\left((t_n:s)+\frac{h}{2}, (\overline{u}_n:s) + \frac{h}{2} f\left((t_n:s),(\overline{u}_n:s)\right)\right)\right)} \right) \end{array} \right).$$

3.2.3.3. Improved Euler's Method based on Contra Harmonic Mean to FIVP

The Improved Euler's method's expected Contra Harmonic mean is precisely defined by

$$\underline{u}_{n+1} = \underline{u}(t_n : s) + F\left(u(t_n : s)\right),$$
$$\overline{u}_{n+1} = \overline{u}(t_n : s) + G\left(u(t_n : s)\right).$$

Where

$$F = \frac{h}{2}\left[\begin{array}{c} f\left(\begin{array}{c} \frac{(t_n:s)^2 + \left((t_n:s)+h\right)^2}{(t_n:s) + \left((t_n:s)+h\right)}, \\ \frac{(\underline{u}_n:s)^2 + \left((\underline{u}_n:s) + \frac{h}{2}\left(f\left((t_n:s),(\underline{u}_n:s)\right) + f\left((t_n:s)+h, (\underline{u}_n:s) + hf\left((t_n:s),(\underline{u}_n:s)\right)\right)\right)\right)^2}{(\underline{u}_n:s) + \left((\underline{u}_n:s) + \frac{h}{2}\left(f\left((t_n:s),(\underline{u}_n:s)\right) + f\left((t_n:s)+h, (\underline{u}_n:s) + hf\left((t_n:s),(\underline{u}_n:s)\right)\right)\right)\right)} \end{array} \right) \\ + f\left(\begin{array}{c} (t_n:s) + h, \\ (\underline{u}_n:s) + h\left(\begin{array}{c} \frac{(t_n:s)^2 + \left((t_n:s)+h\right)^2}{(t_n:s) + \left((t_n:s)+h\right)}, \\ \frac{(\underline{u}_n:s)^2 + \left((\underline{u}_n:s) + \frac{h}{2}\left(f\left((t_n:s),(\underline{u}_n:s)\right) + f\left((t_n:s)+h, (\underline{u}_n:s) + hf\left((t_n:s),(\underline{u}_n:s)\right)\right)\right)\right)^2}{(\underline{u}_n:s) + \left((\underline{u}_n:s) + \frac{h}{2}\left(f\left((t_n:s),(\underline{u}_n:s)\right) + f\left((t_n:s)+h, (\underline{u}_n:s) + hf\left((t_n:s),(\underline{u}_n:s)\right)\right)\right)\right)} \end{array} \right) \end{array} \right) \end{array} \right],$$

$$G = \frac{h}{2}\left[\begin{array}{c} f\left(\begin{array}{c} \frac{(t_n:s)^2 + \left((t_n:s)+h\right)^2}{(t_n:s) + \left((t_n:s)+h\right)}, \\ \frac{(\overline{u}_n:s)^2 + \left((\overline{u}_n:s) + \frac{h}{2}\left(f\left((t_n:s),(\overline{u}_n:s)\right) + f\left((t_n:s)+h, (\overline{u}_n:s) + hf\left((t_n:s),(\overline{u}_n:s)\right)\right)\right)\right)^2}{(\overline{u}_n:s) + \left((\overline{u}_n:s) + \frac{h}{2}\left(f\left((t_n:s),(\overline{u}_n:s)\right) + f\left((t_n:s)+h, (\overline{u}_n:s) + hf\left((t_n:s),(\overline{u}_n:s)\right)\right)\right)\right)} \end{array} \right) \\ + f\left(\begin{array}{c} (t_n:s) + h, \\ (\overline{u}_n:s) + h\left(\begin{array}{c} \frac{(t_n:s)^2 + \left((t_n:s)+h\right)^2}{(t_n:s) + \left((t_n:s)+h\right)}, \\ \frac{(\overline{u}_n:s)^2 + \left((\overline{u}_n:s) + \frac{h}{2}\left(f\left((t_n:s),(\overline{u}_n:s)\right) + f\left((t_n:s)+h, (\overline{u}_n:s) + hf\left((t_n:s),(\overline{u}_n:s)\right)\right)\right)\right)^2}{(\overline{u}_n:s) + \left((\overline{u}_n:s) + \frac{h}{2}\left(f\left((t_n:s),(\overline{u}_n:s)\right) + f\left((t_n:s)+h, (\overline{u}_n:s) + hf\left((t_n:s),(\overline{u}_n:s)\right)\right)\right)\right)} \end{array} \right) \end{array} \right) \end{array} \right].$$

3.2.3.4. Euler's Method based on Centroidal Mean

The Modified Euler's method's expected Centroidal mean is precisely defined by

$$\underline{u}_{n+1} = \underline{u}(t_n : s) + F\left(u(t_n : s)\right),$$
$$\overline{u}_{n+1} = \overline{u}(t_n : s) + G\left(u(t_n : s)\right).$$

Where

$$F = hf\left(\begin{array}{c}\dfrac{2\left((t_n{:}s)^2+(t_n{:}s)\left((t_n{:}s)+h\right)+\left((t_n{:}s)+h\right)^2\right)}{3\left((t_n{:}s)+\left((t_n{:}s)+h\right)\right)},\\ \dfrac{2\left((\underline{u}_n{:}s)^2+(\underline{u}_n{:}s)\left((\underline{u}_n{:}s)+hf\left((t_n{:}s),(\underline{u}_n{:}s)\right)\right)+\left((\underline{u}_n{:}s)+hf\left((t_n{:}s),(\underline{u}_n{:}s)\right)\right)^2\right)}{3\left((\underline{u}_n{:}s)+\left((\underline{u}_n{:}s)+hf\left((t_n{:}s),(\underline{u}_n{:}s)\right)\right)\right)}\end{array}\right),$$

$$G = hf\left(\begin{array}{c}\dfrac{2\left((t_n{:}s)^2+(t_n{:}s)\left((t_n{:}s)+h\right)+\left((t_n{:}s)+h\right)^2\right)}{3\left((t_n{:}s)+\left((t_n{:}s)+h\right)\right)},\\ \dfrac{2\left((\overline{u}_n{:}s)^2+(\overline{u}_n{:}s)\left((\overline{u}_n{:}s)+hf\left((t_n{:}s),(\overline{u}_n{:}s)\right)\right)+\left((\overline{u}_n{:}s)+hf\left((t_n{:}s),(\overline{u}_n{:}s)\right)\right)^2\right)}{3\left((\overline{u}_n{:}s)+\left((\overline{u}_n{:}s)+hf\left((t_n{:}s),(\overline{u}_n{:}s)\right)\right)\right)}\end{array}\right).$$

3.2.3.5. Modified Euler's Method based on Centroidal Mean

The Modified Euler's method's expected Centroidal mean is precisely defined by

$$\underline{u}_{n+1} = \underline{u}(t_n{:}\,s) + F\big(u(t_n{:}\,s)\big),$$
$$\overline{u}_{n+1} = \overline{u}(t_n{:}\,s) + G\big(u(t_n{:}\,s)\big).$$

Where

$$F = hf\left(\begin{array}{c}(t_n{:}\,s)+\frac{h}{2},\\ (\underline{u}_n{:}\,s)+\frac{h}{2}f\left(\begin{array}{c}\dfrac{2\left((t_n{:}s)^2+(t_n{:}s)\left((t_n{:}s)+h\right)+\left((t_n{:}s)+h\right)^2\right)}{3\left((t_n{:}s)+\left((t_n{:}s)+h\right)\right)},\\ \dfrac{2\left((\underline{u}_n{:}s)^2+(\underline{u}_n{:}s)\left((\underline{u}_n{:}s)+hf\left((t_n{:}s)+\frac{h}{2},(\underline{u}_n{:}s)+\frac{h}{2}f\left((t_n{:}s),(\underline{u}_n{:}s)\right)\right)\right)+\left((\underline{u}_n{:}s)+hf\left((\underline{u}_n{:}s)+\frac{h}{2},(\underline{u}_n{:}s)+\frac{h}{2}f\left((t_n{:}s),(\underline{u}_n{:}s)\right)\right)\right)^2\right)}{3\left((\underline{u}_n{:}s)+\left((\underline{u}_n{:}s)+hf\left((t_n{:}s)+\frac{h}{2},(\underline{u}_n{:}s)+\frac{h}{2}f\left((t_n{:}s),(\underline{u}_n{:}s)\right)\right)\right)\right)}\end{array}\right)\end{array}\right),$$

$$G = hf\left(\begin{array}{c}(t_n{:}\,s)+\frac{h}{2},\\ (\overline{u}_n{:}\,s)+\frac{h}{2}f\left(\begin{array}{c}\dfrac{2\left((t_n{:}s)^2+(t_n{:}s)\left((t_n{:}s)+h\right)+\left((t_n{:}s)+h\right)^2\right)}{3\left((t_n{:}s)+\left((t_n{:}s)+h\right)\right)},\\ \dfrac{2\left((\overline{u}_n{:}s)^2+(\overline{u}_n{:}s)\left((\overline{u}_n{:}s)+hf\left((t_n{:}s)+\frac{h}{2},(\overline{u}_n{:}s)+\frac{h}{2}f\left((t_n{:}s),(\overline{u}_n{:}s)\right)\right)\right)+\left((\overline{u}_n{:}s)+hf\left((t_n{:}s)+\frac{h}{2},(\overline{u}_n{:}s)+\frac{h}{2}f\left((t_n{:}s),(\overline{u}_n{:}s)\right)\right)\right)^2\right)}{3\left((\overline{u}_n{:}s)+\left((\overline{u}_n{:}s)+hf\left((t_n{:}s)+\frac{h}{2},(\overline{u}_n{:}s)+\frac{h}{2}f\left((t_n{:}s),(\overline{u}_n{:}s)\right)\right)\right)\right)}\end{array}\right)\end{array}\right).$$

3.2.3.6. Improved Euler's Method based on Centroidal Mean

The Improved Euler's method's expected Centroidal mean is precisely defined by

$$\underline{u}_{n+1} = \underline{u}(t_n{:}\,s) + F\big(u(t_n{:}\,s)\big),$$
$$\overline{u}_{n+1} = \overline{u}(t_n{:}\,s) + G\big(u(t_n{:}\,s)\big).$$

Where

$$F =$$

$$\frac{h}{2}\left[f\left(\frac{2\left((t_n:s)^2+(t_n:s)((t_n:s)+h)+((t_n:s)+h)^2\right)}{3\left((t_n:s)+((t_n:s)+h)\right)}, \frac{2\left((\underline{u}_n:s)^2+(\underline{u}_n:s)\left((\underline{u}_n:s)+\frac{h}{2}\left(f((t_n:s),(\underline{u}_n:s))+f\left((t_n:s)+h,(\underline{u}_n:s)+hf((t_n:s),(\underline{u}_n:s))\right)\right)\right)+\left((\underline{u}_n:s)+\frac{h}{2}\left(f((t_n:s),(\underline{u}_n:s))+f\left((t_n:s)+h,(\underline{u}_n:s)+hf((t_n:s),(\underline{u}_n:s))\right)\right)\right)^2\right)}{3\left((\underline{u}_n:s)+\left((\underline{u}_n:s)+\frac{h}{2}\left(f((t_n:s),(\underline{u}_n:s))+f\left((t_n:s)+h,(\underline{u}_n:s)+hf((t_n:s),(\underline{u}_n:s))\right)\right)\right)\right)}\right)+\right.$$

$$\left.f\left((t_n:s)+h, \frac{2\left((t_n:s)^2+(t_n:s)((t_n:s)+h)+((t_n:s)+h)^2\right)}{3\left((t_n:s)+((t_n:s)+h)\right)}, (\underline{u}_n:s)+h\left(\frac{2\left((\underline{u}_n:s)^2+(\underline{u}_n:s)\left((\underline{u}_n:s)+\frac{h}{2}\left(f((t_n:s),(\underline{u}_n:s))+f\left((t_n:s)+h,(\underline{u}_n:s)+hf((t_n:s),(\underline{u}_n:s))\right)\right)\right)+\left((\underline{u}_n:s)+\frac{h}{2}\left(f((t_n:s),(\underline{u}_n:s))+f\left((t_n:s)+h,(\underline{u}_n:s)+hf((t_n:s),(\underline{u}_n:s))\right)\right)\right)^2\right)}{3\left((\underline{u}_n:s)+\left((\underline{u}_n:s)+\frac{h}{2}\left(f((t_n:s),(\underline{u}_n:s))+f\left((t_n:s)+h,(\underline{u}_n:s)+hf((t_n:s),(\underline{u}_n:s))\right)\right)\right)\right)}\right)\right)\right].$$

$$G =$$

$$\frac{h}{2}\left[f\left(\frac{2\left((t_n:s)^2+(t_n:s)((t_n:s)+h)+((t_n:s)+h)^2\right)}{3\left((t_n:s)+((t_n:s)+h)\right)}, \frac{2\left((\overline{u}_n:s)^2+(\overline{u}_n:s)\left((\overline{u}_n:s)+\frac{h}{2}\left(f((t_n:s),(\overline{u}_n:s))+f\left((t_n:s)+h,(\overline{u}_n:s)+hf((t_n:s),(\overline{u}_n:s))\right)\right)\right)+\left((\overline{u}_n:s)+\frac{h}{2}\left(f((t_n:s),(\overline{u}_n:s))+f\left((t_n:s)+h,(\overline{u}_n:s)+hf((t_n:s),(\overline{u}_n:s))\right)\right)\right)^2\right)}{3\left((\overline{u}_n:s)+\left((\overline{u}_n:s)+\frac{h}{2}\left(f((t_n:s),(\overline{u}_n:s))+f\left((t_n:s)+h,(\overline{u}_n:s)+hf((t_n:s),(\overline{u}_n:s))\right)\right)\right)\right)}\right)+\right.$$

$$\left.f\left((t_n:s)+h, \frac{2\left((t_n:s)^2+(t_n:s)((t_n:s)+h)+((t_n:s)+h)^2\right)}{3\left((t_n:s)+((t_n:s)+h)\right)}, (\overline{u}_n:s)+h\left(\frac{2\left((\overline{u}_n:s)^2+(\overline{u}_n:s)\left((\overline{u}_n:s)+\frac{h}{2}\left(f((t_n:s),(\overline{u}_n:s))+f\left((t_n:s)+h,(\overline{u}_n:s)+hf((t_n:s),(\overline{u}_n:s))\right)\right)\right)+\left((\overline{u}_n:s)+\frac{h}{2}\left(f((t_n:s),(\overline{u}_n:s))+f\left((t_n:s)+h,(\overline{u}_n:s)+hf((t_n:s),(\overline{u}_n:s))\right)\right)\right)^2\right)}{3\left((\overline{u}_n:s)+\left((\overline{u}_n:s)+\frac{h}{2}\left(f((t_n:s),(\overline{u}_n:s))+f\left((t_n:s)+h,(\overline{u}_n:s)+hf((t_n:s),(\overline{u}_n:s))\right)\right)\right)\right)}\right)\right)\right].$$

4. Findings and Discussions with Numerical Illustrations

Think about the differential equation whose fuzzy initial value problem is there.

$$u'(t) = -t^3u,$$
$$u(0) = (0.75 + 0.25s, 1.125 - 0.125s) \text{ Where } 0 \le s \le 1.$$

Solution:

The exact solution is provided by

$$\underline{u}(t_n:s) = \underline{u}(t_n:s)e^{-\frac{t^4}{4}} \text{ and } \overline{u}(t_n:s) = \overline{u}(t_n:s)e^{-\frac{t^4}{4}},$$

After that, the Solutions at $t = 1$,

$$\overline{u}(1:s) = [(0.75 + 0.25s)e^{-0.25}, (1.125 - 0.125s)e^{-0.25}], 0 \le s \le 1.$$

(A) *Enhanced Euler's Methods based on Contra Harmonic Mean and Centroidal Mean*:

Euler's approach provides a precise and fairly accurate result. Below is the Contra Harmonic Mean and Centroidal Mean of the Euler's Method, Modified Euler's Method and Improved Euler's Method using $h = 0.1$:

(A1) *Euler's Method based on Contra Harmonic Mean and Centroidal Mean*:

The approximate solutions obtained by Contra Harmonic Mean and Centroidal Mean of Euler's method for different values of $s \in [0,1]$ when $h = 0.1$ are given below:

Table 1: Numerical Solutions for Euler's Method using Contra Harmonic Mean and Centroidal Mean with $h = 0.1$

s	T	Exact Solutions		Euler's Method		Euler's Method using Contra Harmonic Mean		Euler's Method using Centroidal Mean	
		LOWER	UPPER	Lower	Upper	Lower	Upper	Lower	Upper
0	1	0.58410	0.87615	0.54846	0.82269	0.581800	0.872700	0.58341	0.87512
0.1	1	0.60357	0.86641	0.56674	0.81355	0.601193	0.863003	0.60286	0.86539
0.2	1	0.62304	0.85668	0.58502	0.80441	0.620587	0.853307	0.62231	0.85567
0.3	1	0.64251	0.84694	0.60330	0.79527	0.639980	0.843610	0.64175	0.84595
0.4	1	0.66198	0.83721	0.62159	0.78613	0.659373	0.833913	0.66120	0.83622
0.5	1	0.68145	0.82747	0.63987	0.77698	0.678767	0.824217	0.68065	0.82650
0.6	1	0.70092	0.81774	0.65815	0.76784	0.698160	0.814520	0.70009	0.81678
0.7	1	0.72039	0.80800	0.67643	0.75870	0.717553	0.804823	0.71954	0.80705
0.8	1	0.73986	0.79827	0.69472	0.74956	0.736947	0.795127	0.73899	0.79733
0.9	1	0.75933	0.78853	0.71300	0.74042	0.756340	0.785430	0.75844	0.78761
1	1	0.77880	0.77880	0.73128	0.73128	0.775733	0.775733	0.77788	0.77788

(A2) *Modified Euler Methodbased on Contra Harmonic Mean and Centroidal Mean*:

The approximate solutions obtained by Contra Harmonic Mean and Centroidal Mean of Modified Euler's method for different values of $s \in [0,1]$ when $h = 0.1$ are given below:

Table2: Numerical Solutions for Modified Euler's Method using Contra Harmonic Mean and Centroidal Mean with $h = 0.1$

s	T	Exact Solutions		Euler's Method		Modified Euler's Method using Contra Harmonic Mean		Modified Euler's Method using Centroidal Mean	
		LOWER	UPPER	Lower	Upper	Lower	Upper	Lower	Upper
0	1	0.58410	0.87615	0.54846	0.82269	0.58483	0.87725	0.584812	0.877217
0.1	1	0.60357	0.86641	0.56674	0.81355	0.60432	0.86750	0.604305	0.867471
0.2	1	0.62304	0.85668	0.58502	0.80441	0.62382	0.85775	0.623799	0.857724
0.3	1	0.64251	0.84694	0.60330	0.79527	0.64331	0.84801	0.643293	0.847977
0.4	1	0.66198	0.83721	0.62159	0.78613	0.66281	0.83826	0.662787	0.838230
0.5	1	0.68145	0.82747	0.63987	0.77698	0.68230	0.82851	0.682280	0.828483
0.6	1	0.70092	0.81774	0.65815	0.76784	0.70180	0.81876	0.701774	0.818736
0.7	1	0.72039	0.80800	0.67643	0.75870	0.72129	0.80902	0.721268	0.808989

0.8	**1**	*0.73986*	*0.79827*	*0.69472*	*0.74956*	*0.740790*	*0.79927*	*0.740761*	*0.799243*
0.9	**1**	*0.75933*	*0.78853*	*0.71300*	*0.74042*	*0.760284*	*0.789520*	*0.760255*	*0.789496*
1	**1**	*0.77880*	*0.77880*	*0.73128*	*0.73128*	*0.779779*	*0.77977*	*0.779749*	*0.779749*

(A3) *Improved Euler Method based on Contra Harmonic Mean and Centroidal Mean*:

The approximate solutions obtained by Contra Harmonic Mean and Centroidal Mean of Improved Euler's method for different values of $s \in [0,1]$ when $h = 0.1$ are given below:

Table3: Numerical Solutions for Improved Euler's Method using Contra Harmonic Mean and Centroidal Mean with $h = 0.1$

s	T	Exact Solutions		Euler's Method		Improved Euler's Method using Contra Harmonic Mean		Improved Euler's Method using Centroidal Mean	
		LOWER	UPPER	Lower	Upper	Lower	Upper	Lower	Upper
0	**1**	*0.5841*	*0.8761*	*0.5484*	*0.8226*	*0.570348*	*0.855521*	*0.571118*	*0.856677*
0.1	**1**	*0.6035*	*0.8664*	*0.5667*	*0.8135*	*0.589359*	*0.846016*	*0.590156*	*0.847159*
0.2	**1**	*0.6230*	*0.8566*	*0.5850*	*0.8044*	*0.608371*	*0.836510*	*0.609193*	*0.837640*
0.3	**1**	*0.6425*	*0.8469*	*0.6033*	*0.7952*	*0.627382*	*0.827004*	*0.628230*	*0.828122*
0.4	**1**	*0.6619*	*0.8372*	*0.6215*	*0.7861*	*0.646394*	*0.817498*	*0.647267*	*0.818603*
0.5	**1**	*0.6814*	*0.8274*	*0.6398*	*0.7769*	*0.665406*	*0.807992*	*0.666305*	*0.809084*
0.6	**1**	*0.7009*	*0.8177*	*0.6581*	*0.7678*	*0.684417*	*0.798487*	*0.685342*	*0.799566*
0.7	**1**	*0.7203*	*0.8080*	*0.6764*	*0.7587*	*0.703429*	*0.788981*	*0.704379*	*0.790047*
0.8	**1**	*0.7398*	*0.7982*	*0.6947*	*0.7495*	*0.722440*	*0.779475*	*0.723417*	*0.780528*
0.9	**1**	*0.7593*	*0.7885*	*0.7130*	*0.7404*	*0.741452*	*0.769969*	*0.742454*	*0.771010*
1	**1**	*0.7788*	*0.7788*	*0.7312*	*0.7312*	*0.760463*	*0.760463*	*0.761491*	*0.761491*

(B) *Graphical Representation of Error Analysis for Enhanced Euler's Methods based on Contra Harmonic Mean and Centroidal Mean with* $h = 0.1$ *&* $h = 0.001$:

The Graphical solutions obtained by Contra Harmonic Mean and Centroidal Mean of Enhanced Euler's method for different values of $s \in [0,1]$ with $h = 0.1$ & $h = 0.001$ are given below:

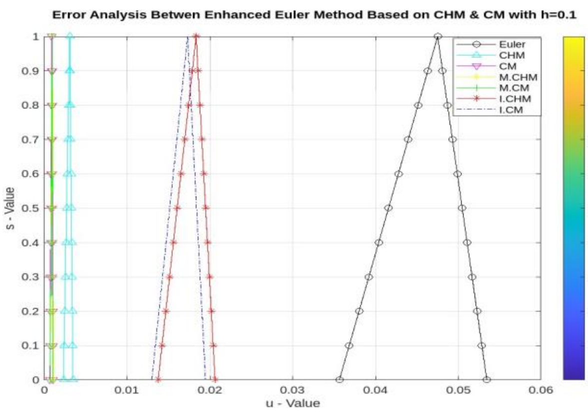

Figure 4: Graphical Representation of Error Analysis for Enhanced Euler's Method using Contra Harmonic Mean and Centroidal Mean with $h = 0.1$

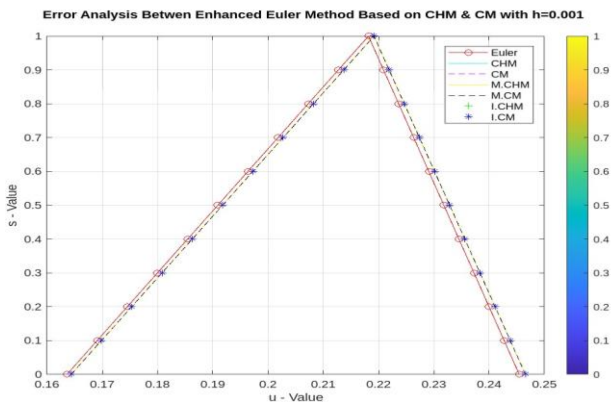

Figure 5: Graphical Representation of Error Analysis for Enhanced Euler's Method using Contra Harmonic Mean and Centroidal Mean with $h = 0.001$

The Fuzzy initial value problem has been solved by proposed enhanced Euler's Methods such as Euler's Method, Modified Euler's Method and Improved Euler's Method using Contra Harmonic Mean and Centroidal Mean. The above tables and figures show the numerical solutions obtained by the proposed methods for different values of $s \in [0,1]$ when $h = 0.1$ and $h = 0.001$. It also shows the solutions obtained by Euler's Method in order to analyse the effectiveness of proposed methods. The analysis concluded that the proposed methods give the very close solutions to the exact values other than Euler's Method.

5. Conclusion:

In order to solve fuzzy initial value problems based on Contra Harmonic Mean and Centroidal Mean, this research proposed three enhanced numerical techniques of Euler, Modified Euler, and Improved Euler. By comparing the example's answers, it was shown that the three enhanced methods outperform the conventional Euler's Method.

References:

[1] Allahviranloo, T., Noeiaghdam, Z., Noeiaghdam, S. & Juan, J.N. A Fuzy Method for Solving Fuzzy Fractional Differential Equation Based on the Generalized Fuzzy Taylor Expansion. Mathematics. 2020; 8(12):1-24. Available from: https://doi.org/10.3390/math8122166.

[2] You., C., Cheng., Y. & Ma, H. Stability of Euler Method for Fuzzy Differential Equation. Symmetry. 2022; 14(16), 1-13. Available from: https://doi.org/10.3390/sym14061279.

[3] Ahmady, N., Allahviranloo, T. &Ahmady, E. A modified Euler method for solving fuzzy differential equations under generalized differentiability. Computational and Applied Mathematics. 2020; 39(2): 1-21. Available from: https://link.springer.com/article/10.1007/s40314-020-1112-1.

[4] Muhammad Imran, Cancan, M., Rashid, S.& Ali, Y. Graphical Comparison on Numerical Solutions of Initial Value Problem by using Euler's method, modified Euler's method and Runge – Kutta method. International Journal of Research Publication and Reviews. 2022; 3(10): 1930-1957. Available from: https://doi.org/10.55248/gengpi.2022.3.10.62.

[5] Md Kamruzzaman & Mithun, C.N. A Comparative Study on Numerical Solution of Initial Value Problem by Using Euler's Method, Modified Euler's Method and Runge – Kutta Method. Journal of Computer and Mathematical Sciences.2018; 9(5), 493-500. Available from: http://dx.doi.org/10.29055/jcms/784.

[6] Pirzada, U.M. & Rama Mohan, S. Fuzzy Form of Euler Method to Solve Fuzzy Differential Equations. Journal of Fuzzy Logic and Modelling in Engineering. 2021; 1(1): 56-62.Available from: https://www.eurekaselect.com/article/113065.

[7] *Ishak, F & Chaini, N.* Numerical computation for solving fuzzy differential equations. The Indonesian Journal of Electrical Engineering and Computer Science (IJEECS). ***2019;*** 16(2): 1026-1033. Available from: http://doi.org/10.11591/ijeecs.v16.i2.pp1026-1033**.**

[8] Abualhomos, M. Numerical Solution of Complex Fuzzy Differential Equations by Euler and Taylor Methods. International Journal of Membrane Science and Technology. 2023;10(3), 340-353. Available from:https://doi.org/10.15379/ijmst.v10i3.1539.

[9] *Balaji, R., AntlineNisha, B., Saradha, M. & Udhayakumar. R.* Numerical Solutions of Fuzzy Differential EquationsBy Harmonic Mean And Cubic Mean of Modified Euler's Method. Contemporary Mathematics. *2023;* 4(3): 581-591. Available from: https://doi.org/10.37256/cm.4320233393.

[10] Ahmady, N., Allahviranloo, T & Ahmady, E. A modified Euler method for solving fuzzy differential equations under generalized differentiability. Computational & Applied Mathematics. 2020; 39(104). Available from: https://link.springer.com/article/10.1007/s40314-020-1112-1

[11] Ahmady, N., Allahviranloo, T & Ahmady, E. An Estimation of the Solution of First Order Fuzzy Differential Equations. Advances in Fuzzy Integral and Differential Equations. 2021;115–141. Available from: https://link.springer.com/chapter/10.1007/978-3-030-73711-5_4

[12] Okeke, A.A & Tumba, P. et al. Analysis and Comparative Study of Numerical Solutions of Initial Value Problems (IVP) in Ordinary Differential Equations (ODE) With Euler and RungeKuttaMethods.American Journal of Engineering Research (AJER). 2019; 8(8): 40-53. Available from: https://www.ajer.org/papers/Vol-8-issue-8/F0808014053.pdf

[13] Workineh, Y., Mekonnen, H & Belew, B. Numerical methods for solving second-order initial value problems of ordinary differential equations with Euler and Runge-Kutta fourth-order methods. Mathematics. 2024. Available from: https://www.semanticscholar.org/paper/Numerical-methods-for-solving-second-order-initial-Workineh-ekonnen/d7d11ba9ba3cbc0ac5f00a619dae987ac3ede9e8